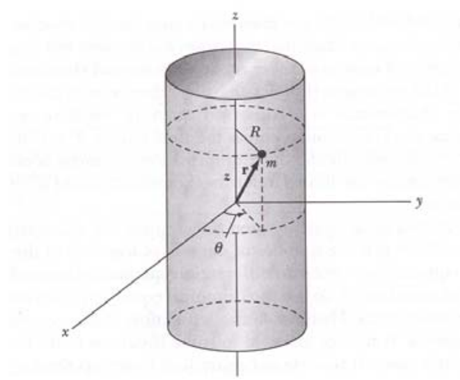
P235 **Practice Midterm Examination** **Prof. Cline**

THIS IS A CLOSED BOOK EXAMINATION.
Do all parts of all four questions.
Show all steps to get full credit.

1:(20pts) Consider a rocket fired vertically upwards from the ground in a uniform vertical gravitation field $g = 9.81 m/s^2$. The rocket, with initial mass m_I, ejects α kg/sec of propellant at an exhaust velocity u m/s. Derive the equation giving the time dependence of the vertical velocity of the rocket.

2; [30pts] Consider a mass m attracted by a force that is directed toward the origin and proportional to the distance from the origin, $\bar{\mathbf{F}} = -k\bar{\mathbf{r}}$. The mass is constrained to move on the surface of a cylinder with the origin on the cylindrical axis as shown. The radius of the cylinder is R.
a) Derive the Lagrangian and Hamiltonian
b) Explain if the Hamiltonian equals the total energy and if it is conserved?
c) Derive the equations of motion
d) Show which variables, if any, are cyclic
e) Derive the frequency of the motion along the axis of the cylinder.



3; (30pts) As part of the Halloween, and the post midterm exam celebrations, Julieta, and Wendi, organize their colleagues to drape crepe ribbons from the highest bridge in Wilson Commons. They plan to do this by dropping rolls of crepe ribbon after fastening the loose end to the parapet. Being very careful physicists they decide that first they need to know the tension in the ribbon to ensure that it does not tear when the roll drops, as well as the speed of the roll after falling $20m$ to ensure it does not hurt anyone. Assume that the rolls of ribbon have a mass M and radius a and that the thickness and mass of the unrolled ribbon is negligible compared with the mass of the roll, that is M and a are constant as it drops, thus the moment of inertia of the roll is $I = \frac{1}{2}Ma^2$.
a) Derive the Lagrangian for one roll of crepe dropping with the upper loose end of the crepe ribbon held fixed.
b) Use Lagrange multipliers to derive the equations of motion
c) Derive the tension in the ribbon.
d) Derive the vertical acceleration and final velocity of the roll after falling $20m$ assuming it starts with zero velocity.

4; [30pts] The P235W midterm exam induces Jacob to try his hand at bungee jumping. Assume Jacob's mass m is suspended in a gravitational field by the bungee of unstretched length b and spring constant k. Besides the longitudinal oscillations due to the bungee jump, Jacob also swings with plane pendulum motion in a vertical plane. Use polar coordinates r, ϕ, neglect air drag, and assume that the bungee always is under tension.
a; Derive the Lagrangian
b; Determine Lagrange's equation of motion for angular motion and identify by name the forces contributing to the angular motion.
c; Determine Lagrange's equation of motion for radial oscillation and identify by name the forces contributing to the tension in the spring.
d; Derive the generalized momenta
e; Determine the Hamiltonian and give all of Hamilton's equations of motion.

Useful formulae

Damped harmonic oscillator:

$$\frac{d^2x}{dt^2} + \Gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

Solution for $\frac{\Gamma}{2} < \omega_o$ is

$$x = Ae^{-\frac{\Gamma t}{2}} \cos(\omega_1 t - \delta)$$

where

$$\omega_1^2 = \omega_0^2 - \frac{\Gamma^2}{4}$$

Sinusoidal-driven damped harmonic oscillator

$$\frac{d^2x}{dt^2} + \Gamma\frac{dx}{dt} + \omega_0^2 x = A\cos(\omega t)$$

Steady-state solution is

$$x_{ss} = \frac{A}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \Gamma^2\omega^2}} \cos(\omega t - \delta)$$

where

$$\tan\delta = \frac{\Gamma\omega}{(\omega_0^2 - \omega^2)}$$

Newton's law of Gravitation

$$\begin{aligned} \overline{\mathbf{F}}_m &= -G\frac{mM}{r^2}\widehat{\mathbf{r}} \\ \overrightarrow{\boldsymbol{\nabla}} \cdot \overrightarrow{\mathbf{g}} &= -4\pi G\rho \end{aligned}$$

Virial Theorem

$$\langle T\rangle = -\frac{1}{2}\left\langle \sum_i \overrightarrow{\mathbf{F}}_i \cdot \overrightarrow{\mathbf{r}}_i \right\rangle$$

Lagrange equations
Generalized coordinates

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

Lagrange equations with undetermined multipliers and generalized forces

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j^{EXC} + \sum_{k=1}^{m} \lambda_k(t)\frac{\partial g_k}{\partial q_j} \qquad (i = 1, 2, 3, ...s)$$

Generalized momentum

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

Hamiltonian

$$H(q_i, p_i, t) = \sum_i p_i\dot{q}_i - L(q_i, \dot{q}_i, t)$$

Hamilton's equations of motion

$$\begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k} \\ -\dot{p}_k &= \frac{\partial H}{\partial q_k} \end{aligned}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$

Routhian reduction

$$\begin{aligned}
R_{cyclic}(q_1,...,q_n;\dot{q}_1,...,\dot{q}_s;p_{s+1},....,p_n;t) &= \sum_{cyclic}^{m} p_i\dot{q}_i - L = H - \sum_{noncyclic}^{s} p_i\dot{q}_i \\
R_{noncyclic}(q_1,...,q_n;p_1,...,p_s;\dot{q}_{s+1},....,\dot{q}_n;t) &= \sum_{noncyclic}^{s} p_i\dot{q}_i - L = H - \sum_{cyclic}^{m} p_i\dot{q}_i
\end{aligned}$$

Cylindrical coordinates

$$\begin{aligned}
L &= T - U \\
&= \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) - U(\rho z\phi)
\end{aligned}$$

$$\begin{aligned}
H &= T + U \\
&= \frac{1}{2m}\left(p_\rho^2 + \frac{p_\phi^2}{\rho^2} + p_z^2\right) + U(\rho, z, \phi)
\end{aligned}$$

Spherical coordinates

$$\begin{aligned}
L &= T - U \\
&= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - U(r\theta\phi)
\end{aligned}$$

$$\begin{aligned}
H &= T + U \\
&= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + U(r, \theta, \phi)
\end{aligned}$$

Orbit differential equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l^2}\frac{1}{u^2}F(\frac{1}{u})$$

Laplace-Runge-Lenz vector

$$\mathbf{A} \equiv (\mathbf{p} \times \mathbf{L}) + (\mu k\hat{\mathbf{r}})$$

Vectors;

$$\overline{\mathbf{C}}=\overline{\mathbf{A}}\times\overline{\mathbf{B}} = \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{A}} = 0$$

$$\overrightarrow{\mathbf{A}} \cdot \left(\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{B}}\right) = 0$$

$$\overrightarrow{\mathbf{A}} \cdot \left(\overrightarrow{\mathbf{B}} \times \overrightarrow{\mathbf{C}}\right) = \left(\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{B}}\right) \cdot \overrightarrow{\mathbf{C}}$$

$$\overrightarrow{\mathbf{A}} \times \left(\overrightarrow{\mathbf{B}} \times \overrightarrow{\mathbf{C}}\right) = \overrightarrow{\mathbf{B}}\left(\overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{C}}\right) - \left(\overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{B}}\right)\overrightarrow{\mathbf{C}}$$

Midterm Examination

Midterm examination:

Will be given Friday 2 November at 1900-2300 hr in **B&L 106.**

Office hours:

Doug Cline; Friday 2 Nov, 1000-1200 hr in B&L203D

Content of midterm:

The midterm examination will comprise four full multipart questions. It is a closed-book, nominally a 3-hour, exam but extra time is scheduled. A calculator may be needed but emphasis will be on concepts and the equations of motion rather than numerical answers. There will be no trick questions, all straightforward bookwork. The questions will emphasize material as discussed below.

Chapters 1, 2: Not explicitly part of the examination although it is assumed that you have a working knowledge of the material in chapter 2.

Chapters 3: Newtonian mechanics is **important**. It will feature throughout the examination implicitly and probably in one question.

Chapter 4: Linear damped oscillations **important** but Fourier analysis will not be emphasized.

Chapter 5: The general features of non-linear systems and chaos should be known, but only qualitative aspects may be addressed in a partial question. Quantitative questions on non-linear systems are not viable on an exam.

Chapter 6: Calculus of variations will not be addressed explicitly, but implicitly it will be assumed in use of Lagrange equations, Hamilton's principle, etc.

Chapter 7: Lagrangian dynamics is **very important**. Applications will cover questions featuring constraints that can be solved with generalized coordinates and Lagrange multipliers. Must know how to use generalized coordinates, derive equations of motion, and derive forces of constraint. Will require ability to solve the equations of motion. You will not be asked to reproduce the derivation of Lagrange equations from either d'Alembert or Hamilton's Principles.

Chapters 8,9: Hamiltonian mechanics is **very important.** Need to know the symmetries/conservation laws, Noether's theorem, cyclic coordinates, properties of Hamiltonian, Hamilton's equations of motion, and Routhian reduction. You should be able to use Hamiltonian mechanics to solve simple mechanical systems.

Chapters 10: Conservative two-body central-force orbits is a **very important** part of the exam. Need to be able to use equations of motion and solve for bound orbits, the Kepler problem, and two-body scattering.

Chapter 11: Non-inertial reference frames will not feature explicitly on the exam, but this topic is included implicitly since it features in rotating systems covered in important aspects of chapters 8 and 9.