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Bell violation for unknown continuous-variable states

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Abstract
We describe a new Bell test for two-particle entangled systems that engages an unbounded continuous variable. The continuous variable state is allowed to be arbitrary and inaccessible to direct measurements. A systematic method is introduced to perform the required measurements indirectly. Our results provide new perspectives on both the study of local realistic theory for continuous-variable systems and on the non-local control theory of quantum information.

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The issue of incompatibility between local realism and the completeness of quantum mechanics was originally raised for unbounded continuous variables in two-party systems by Einstein *et al* [1]. Experiments to test local realism based on inequalities proposed by Bell [2] and his followers [3] imply, as is well known, that classical realism must be discarded as the basis for a universal theory. This has been repeatedly demonstrated in experiments with discrete variable systems [4–8].

Methods for testing local realism in continuous-variable systems have been proposed in order to advance the goal of reaching a completely loophole-free conclusion, and experimental tests on continuous-variable systems have been carried out [9–13]. However, these tests and all continuous-variable proposals to date [9–20] fall short because they rely on advance knowledge

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of the state under test. These methods fail whenever the state under test is unknown because then there is no basis by which measurement strategies can be guaranteed effective. One reason is that non-local correlations present in the original state can evade detection under dimensional reduction [21], as may happen, for example, in pursuing pseudo-spin [13, 18, 19] or binning [20] methods. An exceptional approach by Cavalcanti *et al* [22] leads to a continuous multipartite inequality that does not rely on advance knowledge of the state under test. However, to construct their inequality, operator commutation relations must be ignored, which also eliminates a large category of local realistic theories from test—Sun *et al* [22]. Additionally, violation of these inequalities may not be possible with only two parties—Salles *et al* [22].

Thus two obstacles that have not yet been overcome are these: to derive a standard Bell–Clauser, Horne, Shimony and Holt (CHSH) inequality [3] for an arbitrary and unknown bipartite input state in an unbounded continuous-variable state space, and to describe a currently feasible experimental method for its test. There are significant fundamental and practical reasons for solving this problem. On the fundamental side, a clear understanding of the domains of continuous-variable space which are incompatible with local realism remains to be achieved. More practically, in recent years paradigm-shifting quantum technologies have been developed which depend upon Bell non-locality in theory, and in some cases require the experimental violation of a Bell inequality of an unknown state [23][2]. Methods which permit Bell–CHSH inequalities to be formed and then tested on unknown states in continuous-variable systems may aid in the development and implementation of these technologies.

In this paper we take a significant step towards overcoming both obstacles. To provide easy visualization, we address both issues in a specific scenario using the following two-photon down-conversion state:

$$|\psi_{AB}\rangle = \cos\theta\,|H\rangle_A \otimes \int \mathrm{d}\vec{q}\; h(\vec{q})|\vec{q}\rangle_B + \sin\theta\,|V\rangle_A \otimes \int \mathrm{d}\vec{q}\; v(\vec{q})|\vec{q}\rangle_B, \tag{1}$$

where $|\vec{q}\rangle_B$ is one of a continuum of delta-normalized one-photon transverse momentum states of photon B, and $|H\rangle_A$ and $|V\rangle_A$ denote horizontally and vertically polarized quantum states of photon A. We assume that the transverse momentum state of photon A and the polarization of photon B factor out of the quantum state and therefore need not be indicated.

The $\sin\theta$ and $\cos\theta$ factors are included in writing $|\psi_{AB}\rangle$ to preserve its unit normalization, as the complex continuum amplitudes $h(\vec{q})$ and $v(\vec{q})$ are assumed to be unit-normalized, i.e. $\int \mathrm{d}\vec{q}\; |h(\vec{q})|^2 = \int \mathrm{d}\vec{q}\; |v(\vec{q})|^2 = 1$. Beyond normalization, nothing else is assumed about $h(\vec{q})$ and $v(\vec{q})$, including the value of their generally non-zero scalar product

$$\int \mathrm{d}\vec{q}\; h^*(\vec{q})v(\vec{q}) \equiv z \neq 0. \tag{2}$$

The two-photon state in (1) has an important freedom in the amplitude functions $h(\vec{q})$ and $v(\vec{q})$, which are arbitrary superpositions of the modes in continuous $\vec{q}$ space. In the following we will use the term *bundle* to refer to an arbitrary superposition of $|\vec{q}\rangle$ states. Note that this means that it is impossible to fully determine the state (infinitely many measurements would be required). This point is crucial because it is the stopping point for attempts up to the present time to fully engage a continuous degree of freedom in Bell inequality analysis. We have overcome this roadblock, as we describe below.

[2] These technologies include quantum-assisted communication complexity, quantum-assisted zero-error communication, device-independent quantum key distribution and device-independent randomness generation. For a brief introduction to these topics see Brunner *et al* [23].

It is natural to use the Schmidt analysis in considering two-party pure state entanglement, whether discrete or continuous. The Schmidt decomposition [24] reformulates the state (1) as

$$|\psi_{AB}\rangle = \sum_{n=1}^{\infty} \kappa_n |u_n\rangle_A |f_n\rangle_B, \tag{3}$$

where the sets $\{|u_n\rangle_A\}$ and $\{|f_n\rangle_B\}$ are superpositions of A's polarization states and B's momentum states, respectively, and are derivable as the eigenvectors of A's discrete and B's continuous reduced density matrices. The κ_n^2 are the associated eigenvalues, which are always the same for the two reduced density matrices.

Note that since party A has only two dimensions it has only two eigenvalues, and this forces all but two of B's infinitely many Schmidt eigenvalues to vanish. Thus the infinite n sum in (3) has only two non-zero terms, which we write

$$|\psi\rangle = \kappa_1 |u_1\rangle |f_1\rangle + \kappa_2 |u_2\rangle |f_2\rangle, \tag{4}$$

where we have dropped the A and B labels because it will be easy to remember that the discrete states belong to photon A and the continuous states to photon B. Here $|u_1\rangle$ and $|u_2\rangle$ are merely rotations of the original polarization states $|H\rangle$ and $|V\rangle$, and $|f_1\rangle$ and $|f_2\rangle$ are unknown bundles of B's momentum states $|\vec{q}\rangle$, and we write them as $|f_1\rangle = \int \mathrm{d}\vec{q}\ \varphi_1(\vec{q})|\vec{q}\rangle$ and $|f_2\rangle = \int \mathrm{d}\vec{q}\ \varphi_2(\vec{q})|\vec{q}\rangle$, with a key orthogonality property

$$\langle f_1|f_2\rangle = \int \mathrm{d}\vec{q}\ \varphi_1^*(\vec{q})\varphi_2(\vec{q}) = 0, \tag{5}$$

guaranteed by the Schmidt rearrangement (a similar but fully classical situation has been examined in [25]). κ_1 and κ_2 are real positive coefficients analogous to the $\sin\theta$ and $\cos\theta$ in (1), with $\kappa_1^2 + \kappa_2^2 = 1$. We note that because $h(\vec{q})$ and $v(\vec{q})$ are unknown, then $\{\kappa_1, \kappa_2\}$ are also unknown. Lastly, for simplicity in the following derivation, we assume that z is real-valued which ensures that $|u_i\rangle$ is linearly polarized.

The Schmidt theorem provides an optimum result in three ways. Firstly, as partners for the rotated polarization states it makes two bundles of momentum states from the (presumed unknown) amplitudes $h(\vec{q})$ and $v(\vec{q})$. Secondly, it guarantees that those state bundles are orthogonal, and so we have a pair of orthonormality relations $\langle u_i|u_j\rangle = \langle f_i|f_j\rangle = \delta_{ij}$. Thirdly, independent of the makeup of the two bundles, the Schmidt states $|f_1\rangle$ and $|f_2\rangle$ define a plane in the infinite dimensional $|\vec{q}\rangle$ space.

We are now much closer to Bell inequality territory because rotations in planes in A and B spaces are what the CHSH inequality demands. But the bundles of continuum states making up the two Schmidt states $|f_1\rangle$ and $|f_2\rangle$ are mysterious because the original functions $h(\vec{q})$ and $v(\vec{q})$ were unknown. There are no operators available in continuum B space to make the rotations required by the Bell–CHSH analysis. We will describe below how to make measurements in a rotated basis in the continuum space without rotation operators for the space, but first let us reproduce the Bell–CHSH inequality analysis, under the assumption that rotations in the $|f_1\rangle$–$|f_2\rangle$ plane can be controlled.

With ordinary optical components one can always undertake a rotation of the Schmidt basis in photon A's polarization space, i.e.

$$|u_1^\alpha\rangle = \cos\alpha |u_1\rangle + \sin\alpha |u_2\rangle, \tag{6}$$

$$|u_2^\alpha\rangle = -\sin\alpha |u_1\rangle + \cos\alpha |u_2\rangle, \tag{7}$$

where α defines the arbitrary rotation angle. A rotated basis $|f_1^\beta\rangle$, $|f_2^\beta\rangle$ of momentum space bundles for photon B can be defined similarly with β as the rotation angle in $\vec{q}$ space, while the practical matter of accomplishing such a rotation remains temporarily an open question.

However, given these rotations, the conventional CHSH analysis of local hidden variable theory [3] can be employed. One considers the Bell operator $\mathcal{B}$ and finds $\mathcal{B} \leqslant 2$, where $\mathcal{B}$ is defined as

$$\mathcal{B} = C(\alpha, \beta) - C(\alpha, \beta') + C(\alpha', \beta) + C(\alpha', \beta'). \tag{8}$$

Here $C(\alpha, \beta)$ is the CHSH correlation between photons A and B when the measurements are set for the angles α and β, and $P_{ij}(\alpha, \beta)$ are the joint probabilities of finding photon A in state $|u_i^\alpha\rangle$ and photon B in state $|f_j^\beta\rangle$, with $i, j = 1, 2$. That is,

$$C(\alpha, \beta) = P_{11}(\alpha, \beta) - P_{12}(\alpha, \beta) - P_{21}(\alpha, \beta) + P_{22}(\alpha, \beta). \tag{9}$$

According to quantum mechanics, the joint probability is given as $P_{ij}(\alpha, \beta) = \langle\psi_{AB}|u_i^\alpha\rangle|f_j^\beta\rangle\langle f_j^\beta|\langle u_i^\alpha|\psi_{AB}\rangle$, which is a joint projection in the state spaces of both photons and has the potential to violate the CHSH inequality. Then the Bell operator $\mathcal{B}$ can be calculated to be

$$\begin{aligned}\mathcal{B} = 2\kappa_1\kappa_2 & \left[\sin 2\alpha(\sin 2\beta - \sin 2\beta') + \sin 2\alpha'(\sin 2\beta + \sin 2\beta')\right] \\ & + \cos 2\alpha(\cos 2\beta - \cos 2\beta') + \cos 2\alpha'(\cos 2\beta + \cos 2\beta').\end{aligned} \tag{10}$$

For the choices $\alpha = 0$, $\beta = \pi/8$, $\alpha' = \alpha + \pi/4$ and $\beta' = \beta + \pi/4$, one finds

$$\mathcal{B} = \sqrt{2}(2\kappa_1\kappa_2 + 1). \tag{11}$$

There will be a Bell violation, $\mathcal{B} > 2$, whenever $2\kappa_1\kappa_2 > \sqrt{2} - 1$. Obviously this can be satisfied, and for the state with $\kappa_1 = \kappa_2 = 1/\sqrt{2}$ even the Cirelson bound is attained, i.e. $\mathcal{B}$ reaches the maximum value $2\sqrt{2}$. In fact, as was pointed by Gisin [26], the pure state (4) will always violate the CHSH inequality for any non-zero κ_1 and κ_2 if one chooses the angles α, α', β and β' properly.

As described above, the central hurdle to be overcome is the lack of a method to measure the Schmidt bundles in the continuous $|\vec{q}\rangle$ space of photon B. As we now demonstrate, a specially engineered auxiliary photon is sufficient to accomplish this. The requisite auxiliary photon can be easily created using an auxiliary entangled state which is identical to the original state. Practical techniques for generating pairs of identical entangled biphotons are available, as discussed in the supplementary information (available from stacks.iop.org/NJP/16/013033/mmedia), so we proceed with the setup sketched in figure 1.

Source S_t emits a pair of photons in the desired discrete-continuum entangled state, of which the Schmidt form is

$$|\psi\rangle_{t\bar{t}} = \kappa_1|u_1\rangle_t|f_1\rangle_{\bar{t}} + \kappa_2|u_2\rangle_t|f_2\rangle_{\bar{t}}. \tag{12}$$

The discretely (polarization) entangled photon in mode t is heading northwest (NW) and the continuously (momentum) entangled photon in mode $\bar{t}$ is heading southeast (SE), illustrated by the red paths in figure 1. The goal of our following analysis is to propose a Bell test, namely, measuring various correlations in terms of joint probabilities, for such a discrete-continuum

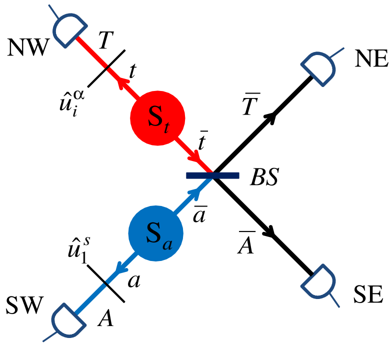

Figure 1. Schematic illustration of Bell test for discrete-continuum entangled photon pairs. The red source S_t emits a photon pair $|\psi\rangle_{t\bar{t}}$, where the discrete (polarization) space of the photon propagating toward NW in mode t is entangled with the continuous (e.g. momentum) space of the photon heading SE in mode $\bar{t}$. The blue source S_a emits identical photon pairs denoted as $|\psi\rangle_{a\bar{a}}$ with the discretely and continuously entangled photons propagating toward SW and NE, respectively. The photon in mode t passes through a polarizer $\hat{u}_i^\alpha$ that passes only the polarization component $|u_i^\alpha\rangle$, and then enters mode T for detection. Similar actions are taken for the photon in mode a with a polarizer $\hat{u}_1^s$ that passes only the polarization component $|u_1^s\rangle$. The photons in modes $\bar{t}$ and $\bar{a}$ are combined by a 50:50 BS with two outcome modes $\bar{T}$ and $\bar{A}$ being detected.

entangled state regardless of what is known or not known about the continuous-space photon in mode $\bar{t}$ and whether it is accessible or not to direct measurement.

A polarization projection on basis $|u_1^\alpha\rangle$ for the photon in mode t can be realized with a polarizer $\hat{u}_1^\alpha$ that passes the $|u_1^\alpha\rangle$ component into mode T, i.e.

$$|u_1^\alpha\rangle_t\, {}_t\langle u_1^\alpha|\psi\rangle_{t\bar{t}} = |u_1^\alpha\rangle_T\{\kappa_1 c_\alpha|f_1\rangle_{\bar{t}} + \kappa_2 s_\alpha|f_2\rangle_{\bar{t}}\}, \tag{13}$$

where c_α and s_α stand for $\cos\alpha$ and $\sin\alpha$. The probability of this measurement outcome being realized is given by $P_1(\alpha) = {}_{t\bar{t}}\langle\psi|u_1^\alpha\rangle_t\, {}_t\langle u_1^\alpha|\psi\rangle_{t\bar{t}} = \kappa_1^2 c_\alpha^2 + \kappa_2^2 s_\alpha^2$, and can be determined experimentally by recording the number of coincidences detected during a fixed time window in modes $(T, \bar{t})$ and $(t, \bar{t})$ for polarizer angle α

$$P_1(\alpha) = \frac{N_\alpha(T, \bar{t})}{N(t, \bar{t})}, \tag{14}$$

where $N_\alpha(T, \bar{t})$ and $N(t, \bar{t})$ are the number of coincidences in their corresponding modes. This also gives the value of κ_1 and κ_2 since $\kappa_1^2 + \kappa_2^2 = 1$ as stated after (5).

To determine joint probabilities, one needs to measure the continuum space in a basis rotated by the angle β as well, so we now express the state in the rotated basis, $\{|f_1^\beta\rangle, |f_2^\beta\rangle\}$,

$$|u_1^\alpha\rangle_t\, {}_t\langle u_1^\alpha|\psi\rangle_{t\bar{t}} = |u_1^\alpha\rangle_T\left\{(\kappa_1 c_\alpha c_\beta + \kappa_2 s_\alpha s_\beta)|f_1^\beta\rangle_{\bar{t}} + (-\kappa_1 c_\alpha s_\beta + \kappa_2 s_\alpha c_\beta)|f_2^\beta\rangle_{\bar{t}}\right\}, \tag{15}$$

which we rewrite again as

$$|u_1^\alpha\rangle_t\, {}_t\langle u_1^\alpha|\psi\rangle_{t\bar{t}} = \sqrt{P_1(\alpha)}|u_1^\alpha\rangle_T\left(c_{11}|f_1^\beta\rangle_{\bar{t}} + c_{12}|f_2^\beta\rangle_{\bar{t}}\right). \tag{16}$$

Here c_{ij} with $i, j = 1, 2$, are normalized amplitude coefficients, and they relate to joint probabilities in an obvious way: $P_{ij}(\alpha, \beta) = |c_{ij}|^2 P_i(\alpha)$.

Now that the probability $P_i(\alpha)$ can be measured easily, as is shown above, the value of joint probability $P_{ij}(\alpha, \beta)$ can be determined by measuring only the coefficients $|c_{ij}|^2$. This can be realized with the help of the auxiliary photon pair $|\psi\rangle_{a\bar{a}}$, which is generated by source S_a to have exactly the same form as the state under test, i.e.

$$|\psi\rangle_{a\bar{a}} = \kappa_1 |u_1\rangle_a |f_1\rangle_{\bar{a}} + \kappa_2 |u_2\rangle_a |f_2\rangle_{\bar{a}} \tag{17}$$

with the discretely entangled photon in mode a heading SW and the continuously entangled photon in mode $\bar{a}$ heading NE, illustrated by the blue paths in figure 1.

The auxiliary photon pair allows us to perform an indirect measurement in the continuous-variable space of the photon in mode $\bar{t}$. First, the mode a photon of the auxiliary pair is projected (by a polarizer $\hat{u}_1^s$) onto the polarization basis $|u_1^s\rangle$, where angle s is chosen to strip off the $|f_2^\beta\rangle$ component from the photon in mode $\bar{a}$. A glance at (15) shows how a stripping in continuum space by action in polarization space works. In (15), by choosing α such that $\kappa_1 \tan\beta = \kappa_2 \tan\alpha$, the $|f_2^\beta\rangle$ component would be eliminated. In the case of auxiliary photon a, we choose s such that $\kappa_1 \tan\beta = \kappa_2 \tan s$ and obtain

$$|u_1^s\rangle_{a\,a}\langle u_1^s|\psi\rangle_{a\bar{a}} = \sqrt{P_1(s)}|u_1^s\rangle_A |f_1^\beta\rangle_{\bar{a}} \tag{18}$$

with $P_1(s) = {}_{a\bar{a}}\langle\psi|u_1^s\rangle_{a\,a}\langle u_1^s|\psi\rangle_{a\bar{a}}$. $P_i(s)$ is determined experimentally in exactly the same way as $P_i(\alpha)$. The photon enters mode A from mode a after passing the stripping polarizer $\hat{u}_1^s$, as shown in figure 1. Then the four-photon state after the two polarization projections in modes t and a is given by

$$|\psi\rangle_{T\bar{t}A\bar{a}} = \sqrt{P_1(\alpha)P_1(s)}|u_1^\alpha\rangle_T |u_1^s\rangle_A \otimes \left(c_{11}|f_1^\beta\rangle_{\bar{t}} + c_{12}|f_2^\beta\rangle_{\bar{t}}\right) \otimes |f_1^\beta\rangle_{\bar{a}}. \tag{19}$$

Next, as shown in figure 1, the mode $\bar{t}$ photon is combined with the mode $\bar{a}$ photon (which is in the continuous variable state $|f_1^\beta\rangle$) by a 50:50 beam splitter (BS). The outcome modes are denoted as $\bar{T}$ (NE) and $\bar{A}$ (SE). The effect of the BS can be expressed as

$$|f_j^\beta\rangle_{\bar{t}} = \left(|f_j^\beta\rangle_{\bar{A}} + i|f_j^\beta\rangle_{\bar{T}}\right)/\sqrt{2}, \tag{20}$$

$$|f_j^\beta\rangle_{\bar{a}} = \left(i|f_j^\beta\rangle_{\bar{A}} + |f_j^\beta\rangle_{\bar{T}}\right)/\sqrt{2}, \tag{21}$$

where $j = 1, 2$. As a result of Hong–Ou–Mandel interference [27], the coincidence of the outcome photons in modes $\bar{T}$ and $\bar{A}$ determines the degree of distinguishability between the photons in modes $\bar{t}$ and $\bar{a}$. To be more specific, the contributing component of the mode $\bar{t}$ photon in equation (19) to the coincidences after the BS is $c_{12}|f_2^\beta\rangle_{\bar{t}}$, which is the distinguishable component of the photon in mode $\bar{a}$. This amounts to a filtering or projecting operation of the photon in mode $\bar{t}$ onto the continuous variable basis $|f_2^\beta\rangle$.

With the above operations, a joint projection is realized for testing the entangled photon pair $|\psi\rangle_{t\bar{t}}$. It is then straightforward to achieve the joint probability $P_{12}(\alpha, \beta)$. The four-photon coincidence probability in modes $T, \bar{T}, A, \bar{A}$ is given as

$$P_{T\bar{T}A\bar{A}}(\alpha, \beta) = \frac{P_1(\alpha)P_1(s)|c_{12}|^2}{2} = \frac{N_{\alpha\beta}(T, A, \bar{T}, \bar{A})}{N(t, a, \bar{t}, \bar{a})}, \tag{22}$$

where $N_{\alpha\beta}(T, A, \bar{T}, \bar{A})$ and $N(t, a, \bar{t}, \bar{a})$ are four-photon coincidence counts of the corresponding modes for polarization angles α and β. The individual probabilities can be determined using (14). Consequently, the joint probability can be written in terms of measurable quantities

$$P_{12}(\alpha, \beta) = \frac{2N(a, \bar{a})N_{\alpha\beta}(T, A, \bar{T}, \bar{A})}{N_s(A, \bar{a})N(t, a, \bar{t}, \bar{a})}. \tag{23}$$

Measurement of the other joint probabilities $P_{11}(\alpha, \beta)$ and $P_{2j}(\alpha, \beta)$ are accomplished by appropriately rotating the angles α and β by $\pi/2$. In this way the correlation function $C(\alpha, \beta)$ can be achieved straightforwardly. Other correlations can be obtained similarly with other choices of angles α and β. To achieve the Bell violation given in (11) the orientation of the stripping polarizer is determined as $\tan s = (\kappa_1/\kappa_2)(\sqrt{2}-1)$ and $\tan s' = (\kappa_1/\kappa_2)(\sqrt{2}+1)$.

Beyond the Bell violation issue, it is important to note that our method of measuring the continuous-variable space is an example of non-local quantum control and is essentially the same as that used by Pavičić and Summhammer [28] in an early entanglement swapping experiment. It provides a new perspective on indirect measurement of a system state which is not directly accessible experimentally. We have shown explicitly how, by manipulating a discrete and controllable entangled partner, measurements of a continuum system may be made. Apart from increased measurement capabilities, this type of indirect measurement may be useful for transferring or encoding information into continuous-variable spaces which are difficult to detect or probe directly. Therefore, with proper design, it may be possible to construct communication protocols which impede potential eavesdroppers from obtaining the encoded information.

In summary, we have addressed the two obstacles mentioned in paragraph 3, obtaining a resolution with the aid of a new approach to continuous-variable measurement. Specifically, we have devised a Bell-CHSH inequality for the two-particle case in which one particle is defined by an unbounded continuous variable in a unknown state of arbitrary complexity, and we have sketched a currently feasible measurement approach for its implementation. This technique may expand further the systems in which Bell non-locality may be used for practical applications [23].

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