PHY 415 E&M
Homework 1

1. (1)

$$
\begin{aligned}
& \nabla \times (\nabla \times \vec{A}) \\
= \;& \varepsilon_{ijk}\partial_j(\varepsilon_{klm}\partial_l A_m) \\
= \;& \varepsilon_{kij}\varepsilon_{klm}\partial_i\partial_l A_m \\
= \;& (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l A_m \\
= \;& \partial_i\partial_m A_m - \partial_j\partial_j A_m \\
= \;& \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}
\end{aligned}
$$

□

(2)

$$
\begin{aligned}
& \nabla \cdot (\vec{A} \times \vec{B}) \\
= \;& \partial_k \varepsilon_{lmk} A_l B_m \\
= \;& \varepsilon_{lmk}(\partial_k A_l)B_m + \varepsilon_{lmk} A_l (\partial_k B_m) \\
= \;& (\varepsilon_{mkl}\partial_k A_l)B_m + A_l(-\varepsilon_{lkm}\partial_k B_m) \\
= \;& \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})
\end{aligned}
$$

□

(3)

$$
\nabla \cdot \vec{r} = \sum_{i=1}^{3} \partial_i r_i = 1 + 1 + 1 = 3
$$

□

(4) (i) $\vec{r} \neq 0$

$$
\begin{aligned}
\nabla^2 \left(\frac{1}{|\vec{r}|}\right) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\frac{1}{|\vec{r}|}\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\cancel{r^2} \cdot \frac{-1}{\cancel{r^2}}\right) \\
&= 0
\end{aligned}
$$

(ii) $\vec{r} = 0$

$$\begin{aligned}
\int \nabla^2 \left(\frac{1}{|\vec{r}|}\right) d\tau &= \int \nabla \cdot \nabla \left(\frac{1}{|\vec{r}|}\right) d\tau \\
&= \oint_s \nabla \left(\frac{1}{|\vec{r}|}\right) \cdot d\vec{\sigma} \\
&= \int -\frac{\hat{r}}{r^2} \cdot d\vec{\sigma} \\
&= \int -\frac{\hat{r}}{\cancel{r^2}} \cdot \cancel{r^2} \sin\theta d\theta d\phi \hat{r} \\
&= -4\pi \int \delta^3(\vec{r}) d\tau
\end{aligned}$$

$$\therefore \nabla \left(\frac{1}{|\vec{r}|}\right) = -4\pi\delta^3(\vec{r})$$

□

2.

$$\begin{aligned}
\vec{E} &= -\nabla\phi(\vec{r}) \\
&= -q\left[-\frac{e^{-\mu r}}{r^2} - \frac{\mu e^{-\mu r}}{r}\right]\hat{r} \\
&= \frac{qe^{-\mu r}}{r^2}[1+\mu r]\hat{r}
\end{aligned}$$

$$\begin{aligned}
\oint \vec{E} \cdot d\vec{\sigma} &= q \oint \frac{e^{-\mu r}}{\cancel{r^2}}[1+\mu r] \cdot \cancel{r^2} \sin\theta d\theta d\phi \\
&= qe^{-\mu r}[1+\mu r]4\pi \\
&= 4\pi q e^{\mu r}[1+\mu r] \neq 4\pi q
\end{aligned}$$

$\longrightarrow$ Gauss' law invalid.

□

$$\begin{aligned}
\nabla^2\phi &= \nabla \cdot (\nabla\phi) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}[-qe^{-\mu r}(1+\mu r)] \\
&= \frac{-1}{r^2} q[-\mu e^{-\mu r}(\cancel{1}+\mu r) + \cancel{\mu e^{-\mu r}}] \\
&= \frac{q}{r^2}\mu^2 r e^{-\mu r} \\
&= \mu^2 \frac{q}{r} e^{-\mu r} \\
&= \mu^2\phi
\end{aligned}$$

□

3. Gauss' Law

$$\oint \vec{E}\cdot d\vec{\sigma} = 4\pi Q$$

(i) $r < R$

$$\begin{aligned}
\cancel{4\pi}\cancel{r^2}E_r &= \cancel{4\pi}\frac{4\pi}{3}r^{\cancel{3}}\rho \\
E_r &= \frac{4\pi}{3}\rho r \\
\vec{E}(r) &= \frac{4\pi}{3}\rho r\hat{r}
\end{aligned}$$

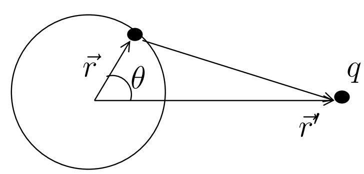
□

(ii) $r > R$

$$\begin{aligned}
\cancel{4\pi}r^2E_r &= \cancel{4\pi}\frac{4\pi}{3}R^3\rho \\
E_r &= \frac{4\pi}{3}\frac{R^3}{r^2}\rho \\
\vec{E}(r) &= \frac{4\pi}{3}\frac{R^3}{r^2}\rho\hat{r}
\end{aligned}$$

□

4.



$$\begin{aligned}
\oint \vec{E}\cdot d\vec{\sigma} &= 2\pi R^2\int_0^{\pi}\frac{q}{(r^2+R^2-2rR\cos\theta)^{\frac{1}{2}}}\cdot\frac{(\vec{r}-\vec{r'})\cdot\hat{r}}{r^2+R^2-2rR\cos\theta}\sin\theta d\theta \\
&= 2\pi R^2 q\int_{-1}^{1}\frac{R-r\cos\theta}{(r^2+R^2-2rR\cos\theta)^{\frac{3}{2}}}d\cos\theta \\
&= 2\pi R^2 q\{\int_{-1}^{1}\frac{R}{(r^2+R^2-2rRx)^{\frac{3}{2}}}dx - \int_{-1}^{1}\frac{rx}{(r^2+R^2-2rRx)^{\frac{3}{2}}}dx\} \\
&= 2\pi R^2 q\{\frac{1}{r(r^2+R^2-2rRx)^{\frac{1}{2}}}|_{-1}^{1} - r\left[\frac{x}{Rr(r^2+R^2-2rRx)^{\frac{1}{2}}}|_{-1}^{1}\right. \\
&\quad \left.-\int_{-1}^{1}\frac{dx}{Rr(r^2+R^2-2rRx)^{\frac{1}{2}}}\right]\} \\
&= 0
\end{aligned}$$

□