PHY 415
Homework 4
Solutions

Problem 1

(a) $\phi(\vec{r}) = \int \frac{\rho(\vec{r'})}{|\vec{r}-\vec{r'}|} d^3r'$

$= \int \rho(\vec{r'}) \sum_{n=0}^{\infty} P_n(\cos\gamma) \frac{r'^n}{r^{n+1}} d^3r'$

$= \int d^3r' \rho [P_0 \frac{1}{r} + P_1 \frac{r'}{r^2} + P_2 \frac{r'^2}{r^3} + ...]$

$= \int d^3r' \frac{\rho(\vec{r'})}{r} [1 + \cos\gamma \frac{r'}{r} + \frac{1}{2}(3\cos^2\gamma - 1)(\frac{r'}{r})^2 + ...]$

where

$\int d^3r' \frac{\rho(\vec{r'})}{r}$

$= \int d^3r' \frac{1}{r}(r'^2 e^{-r'} \sin^2\theta') r'^2 \sin\theta' d\theta' d\phi' dr'$

$= \frac{2\pi}{r}(4!) \int_{-1}^{1} 1 - \cos^2\theta' d\cos\theta' = \frac{64\pi}{r}$

$\int d^3r' \frac{\rho(\vec{r'})}{r} \cos\gamma \frac{r'}{r} = 0$

$\int d^3r' \frac{\rho(\vec{r'})}{r} \frac{1}{2}(3\cos^2\gamma - 1)(\frac{r'}{r})^2$

$= \frac{1}{2r^3} \int r'^4 e^{-r'} dr' \sin^3\theta' [3[\cos^2\theta \cos^2\theta' + (\sin\theta \sin\theta' \cos(\phi-\phi'))^2 + 2\cos\theta\cos\theta' \sin\theta \sin\theta' \cos(\phi-\phi')] - 1]$

$= \frac{64\pi 3(1-3\cos^2\theta)}{r^3}$

$\therefore \phi(\vec{r'})$ for $r \Rightarrow \infty \sim \frac{64\pi}{r}(1 + \frac{3}{r^2}(1 - 3\cos^2\theta))$

(b)

The multipole moment

(i) monopole, $\phi = \frac{64\pi}{r} \Rightarrow q = 64\pi$

(ii) dipole, $\phi = 0$

(iii) quadrupole, $\phi = \frac{64\pi}{r} \frac{3(1-3\cos^2\theta)}{r^2}$

$Q_{ij} = \int (3x'_i x'_j - r^2 \delta_{ij}) \rho(\vec{r'}) d^3r'$

$$\Rightarrow Q = \begin{pmatrix} 384\pi & 0 & 0 \\ 0 & 384\pi & 0 \\ 0 & 0 & -768\pi \end{pmatrix}$$

Problem 2

(a)

$r > b, \nabla \cdot \vec{E} = 4\pi\rho$

$\Rightarrow \vec{E} = \frac{q}{r^2}\hat{r}$

$\Rightarrow \phi(r) = \frac{q}{r}$

(b) $\phi(r)$ should be constant inside the conductor

$\phi(a) = \phi(b) = -\int_{\infty}^{b} \frac{q}{r^2} dr = \frac{q}{b}$

(c)

(i)potential remains the same inside the conductor $\phi(r < a < b) = \frac{q}{b}$

(ii)the potential $\phi_{induced}(r < a)$ due to induced charge distribution could be viewed as the potential caused by an image charge q' at d'

$q' = -\frac{a}{a/2} = -2q, d = \frac{a^2}{a/2} = 2a$

$\therefore \phi(0) = \frac{q}{b} + \frac{q}{a/2} - \frac{2q}{2a} = q(\frac{1}{b} + \frac{1}{a})$

Problem 3

$\Phi_{real} = \frac{q}{\sqrt{r^2+(z'-z)^2}}$

$\Phi_{total} = \sum_0^\infty [\frac{q}{\sqrt{r^2+(2dn+a-z)^2}} - \frac{q}{\sqrt{r^2+(2dn-a-z)^2}}]$

$\sigma = \frac{1}{4\pi}\hat{z} \cdot (-\nabla\phi)$

$\nabla\phi = \frac{\delta}{\delta z}\Phi$

$= q\sum_0^\infty [\frac{2dn+a-z}{(r^2+(2dn+a-z)^2)^{3/2}} - \frac{2dn-a-z}{(r^2+(2dn-a-z)^2)^{3/2}}]$

$\Rightarrow$

$\sigma(z=0) = \frac{q}{4\pi}\sum_0 \infty [\frac{2dn-a}{(r^2+(2dn-a)^2)^{3/2}} - \frac{2dn+a}{(r^2+(2dn+a)^2)^{3/2}}]$

$\sigma(z=d) = \frac{q}{4\pi} \sum_0 \infty [\frac{2dn-a-d}{(r^2+(2dn-a-d)^2)^{3/2}} - \frac{2dn+a-d}{(r^2+(2dn+a-d)^2)^{3/2}}]$

$\Rightarrow$

$Q_0 = \int \sigma_0 2\pi r dr + \int \sigma_d 2\pi r dr = -q$