PHY 415
Homework 5
Solutions

1.

$$
\begin{aligned}
\nabla^2\phi(\vec{r}) &= 0 \\
\phi(\vec{r}) &= \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-(l+1)})P_l(\cos\theta)
\end{aligned}
$$

(i) $r < R$

$$
\begin{aligned}
\phi(\vec{r}) &= \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \\
\Phi_0\cos\theta &= \sum A_l R^l P_l(\cos\theta) \\
A_{l\neq 1} &= 0 \\
A_1 &= \Phi_0\frac{1}{R} \\
\phi_{in}(\vec{r}) &= \Phi_0\frac{r}{R}\cos\theta
\end{aligned}
$$

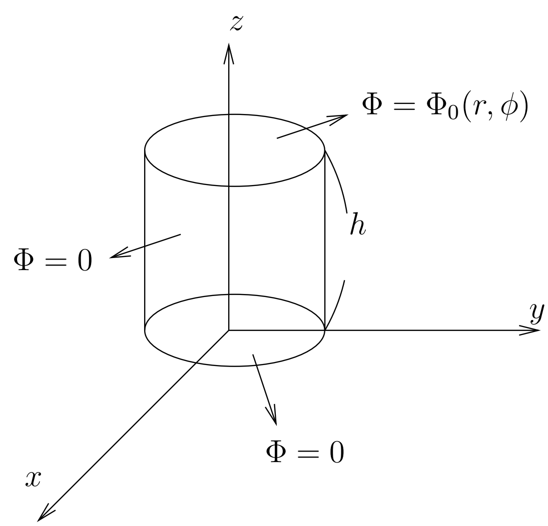
□

(ii) $r > R, A_l = 0$

$$
\begin{aligned}
\phi(\vec{r}) &= \sum B_l\frac{1}{r^{l+1}}P_l(\cos\theta) \\
\Phi_0\cos\theta &= \sum B_l\frac{1}{R^{l+1}}P_l(\cos\theta) \\
\Phi_0\cancel{\cos\theta} = B_l &= B_1\frac{1}{R^2}\cancel{P_1(\cos\theta)} \\
B_1 &= R^2\Phi_0 \\
\therefore \Phi_{out}(\vec{r}) &= \Phi_0\left(\frac{R}{r}\right)^2\cos\theta
\end{aligned}
$$

□

2.



$\Phi(r,\phi,z) = R(r)\phi(\phi)Z(z)$ satisfy $\nabla^2\Phi = 0$.

$$\Rightarrow \quad \frac{\partial^2}{\partial r^2}\Phi + \frac{1}{r}\frac{\partial}{\partial r}\Phi + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

$$\Rightarrow \quad \left\{ \begin{array}{lcl} \dfrac{d^2Z}{dz^2} - k^2 Z & = & 0 \\ \dfrac{d^2\phi}{d\phi^2} + \nu^2\phi & = & 0 \\ \dfrac{d^2R}{dr^2} + \dfrac{1}{r}\dfrac{dR}{dr} + (K^2 - \dfrac{\nu^2}{r^2})R & = & 0 \end{array} \right.$$

$$\left\{ \begin{array}{lcl} Z(z) & = & A\sinh kz + B\cosh kz \\ \phi(\phi) & = & C\sin m\phi + D\cos m\phi \\ R(r) & = & EJ_m(K_{mn}r) + FN_m(K_{mn}r) \end{array} \right.$$

$B = 0, \because \Phi(r,\phi,0) = 0$

$G = 0, \because \Phi(0,\phi,z)$ finite.

$$\therefore \phi(r,\phi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(K_{mn}r)\sinh(k_{mn}Z)\cdot(C_{mn}\sin m\phi + D_{mn}\cos m\phi)$$

$$C_{mn} = \frac{2\,\mathrm{csch}(K_{mn}h)}{\pi R^2 J_{m+1}^2(K_{mn}a)}\int_0^{2\pi} d\phi \int_0^R dr\Phi(r,\phi)J_m(K_{mn}r)\sin m\phi$$

$$D_{mn} = \frac{2\,\mathrm{csch}(K_{mn}h)}{\pi R^2 J_{m+1}^2(K_{mn}a)}\int_0^{2\pi} d\phi \int_0^R dr\Phi(r,\phi)J_m(K_{mn}r)\cos m\phi$$

□

3. (a) From Green's Theorem

$$\int_r d^3r(\phi_1\nabla^2\phi_2 - \phi_2\nabla^2\phi_1) = \int_s ds\cdot(\phi_1\nabla\phi_2 - \phi_2\nabla\phi_1) \qquad \text{(eq. 3.68)}$$

(i) $\nabla^2\phi_1 = -4\pi\rho_1, \nabla^2\phi_2 = -4\pi\rho_2$

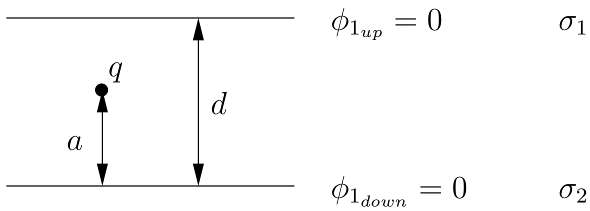
(ii) $\nabla\phi_1 = 4\pi\sigma_1, \nabla\phi_2 = 4\pi\sigma_2$

$\Rightarrow -\cancel{4\pi}\int d^3r(\phi_1\rho_2 - \phi_2\rho_1) = \cancel{4\pi}\int(\phi_1\sigma_2 - \phi_2\sigma_1)ds$

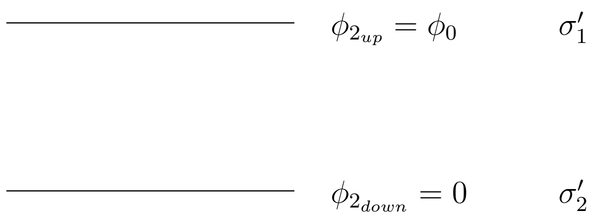
$\Rightarrow \int d^3r\phi_1\rho_2 + \int_s ds\phi_1\sigma_2 = \int d^3r\phi_2\rho_1 + \int_s ds\phi_2\sigma_1$

□

(b) Condition I



Condition II



From reciprocity theorem:

$\int q\delta(z-a)\frac{a}{d}\phi_0 d^3r + \int\sigma_1\phi_0 ds + \int\sigma_2 0 ds = \int 0\psi_1 d^3r + \int\sigma_1' 0 ds + \int\sigma_2' 0 ds$

$\Rightarrow q\frac{a}{d}\cancel{\phi_0} = -\cancel{\phi_0}\int\sigma_1 ds$

$\Rightarrow \sigma_1 = -q\frac{a}{d}$

□

Same, we can find

$$\sigma_2 = -q\frac{d-a}{d}$$

□