Finite Temperature Effective Actions

Ashok Das[a,b] and J. Frenkel[c*]

[a] *Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627-0171, USA*

[b] *Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Calcutta 700064, India and*

[c] *Instituto de Física, Universidade de São Paulo, 05508-090, São Paulo, SP, BRAZIL*

Within the context of non-Hermitian quantum mechanics, we use the generators of eigenvectors of the Hamiltonian to construct a unitary inner product space. Such models have been of interest in recent years, for instance, in the context of $\mathcal{PT}$ symmetry, although our construction extends to the larger class of so-called pseudo-Hermitian Operators. We provide a detailed example to illustrate the concept and compare with known results.

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Effective action for a system interacting with a background field is an important fundamental concept in quantum field theory which incorporates all the one loop quantum mechanical corrections to a theory. At zero temperature we know that the n-point amplitudes (involving the background fields) at one loop are divergent and, consequently, the evaluation of the effective action at $T=0$ needs a regularization. Effective actions can, of course, be evaluated perturbatively. However, a beautiful method due to Schwinger [1], also known as the proper time formalism, is quite useful in evaluating one loop effective actions at zero temperature with a gauge invariant regularization (in the case of gauge backgrounds). It involves solving dynamical equations in proper time, which are not always trivial, to determine th effective action. When the dynamical equations can be solved in a closed form, the gauge invariant regularized action can be given a closed form or at least an integral representation. This has been profitably used to calculate the imaginary part of the effective action for fermions interacting with a constant background electromagnetic field which describes the decay rate of the vacuum [1]. However, when the dynamical equations cannot be solved in a closed form, the method leads to a perturbative determination of the effective action.

In the past couple of decades, there have been several attempts [2] to generalize the method due to Schwinger to finite temperature [3, 4] and to determine the imaginary part of the effective action leading to conflicting results. We believe that since, unlike the zero temperature amplitudes, the amplitudes at finite temperature are ultraviolet finite, it is not essential to generalize the method due to Schwinger to finite temperature. After all, the proper time method was designed to provide a (gauge invariant) regularization which is not necessary at finite temperature. Therefore, we propose an alternative direct method for evaluating finite temperature effectve actions based mainly on the properties of systems at finite temperature. In this connection, we believe that the real time formalism [4] (we use the closed time path formalism due to Schwinger [5]) is more suited for this purpose. We note that, in general, the imaginary time formalism (the Matsubara formalism [6]) leads naturally to retarded and advanced amplitudes, but the derivation of Feynman amplitudes (beyond the two point function) in this formalism still remains an open question. On the other hand, the effective action that we are interested in is precisely the one that generates Feynman amplitudes. In contrast, the effective action, when evaluated properly in the real time formalism, leads naturally not only to the Feynman amplitudes, but also to the retarded as well as the advanced amplitudes as we will show in examples. Furthermore, as we have emphasized earlier in [4, 7], the real time calculations can be carried out quite easily in the mixed space where the spatial coordinates have been Fourier transformed as we will describe in the particular examples.

Let us consider a system of fermions interacting with an external which we generically denote by A. This can be a scalar or a vector background and we suppress the Lorentz index (structure) of the background field for simplicity. If the fermion has a mass m, it is straightforward to obtain from the definition of the effective action that

$$\frac{\partial\Gamma_{\rm eff}}{\partial m} = \int {\rm d}t{\rm d}\mathbf{x}\ S(t,\mathbf{x};t,\mathbf{x}), \tag{1}$$

where $S(t,\mathbf{x};t',\mathbf{x}')$ denotes the complete fermion propagator in the presence of the background field. However, keeping in mind that the fermion may not always have a mass (say, for example, in the Schwinger model [8]), we use alternatively the fact that the variation of the effective action with respect to the background field leads to the generalized fermion "propagator" at coincident points (even though we are using the same symbol as in (1) the exact meaning of S below depends on the nature of the background field as we explain below),

$$\frac{\delta\Gamma_{\rm eff}}{\delta A(t,\mathbf{x})} = S(t,\mathbf{x};t,\mathbf{x}), \tag{2}$$

where we are suppressing the Lorentz structure of the background field as well as of the generalized "propagator". We note that for a scalar background, the right hand side of (2) indeed denotes the complete fermion

* e-mail: das@pas.rochester.edu, jfrenkel@fma.if.usp.br

propagator of the interacting theory at coincident coordinates. On the other hand, for a gauge field background, the right hand side determines the current density of the theory which is related to the complete fermion propagator of the theory through a Dirac trace involving the Dirac matrix. In either case, we note that it is the fermion propagator that is relevant in (2) in the evaluation of the effective action. In the mixed space (where the coordinates $\mathbf{x}$ have been Fourier transformed), we can write (2) as

$$\frac{\delta\Gamma_{\rm eff}}{\delta A(t,-\mathbf{p})} = S(t,t;\mathbf{p}). \tag{3}$$

Since the effective action is so intimately connected with a determination of the fermion propagator and since we are not interested in the zero temperature part of the effective action, our proposal is to determine the complete fermion propagator at finite temperature directly such that

(i) it satisfies the appropriate equation for the complete Green's function of the theory,

(ii) it satisfies the necessary symmetry properties of the theory such as the Ward identity,

(iii) and most importantly, it satisfies the anti-periodicity property associated with a finite temperature fermion propagator.

It is, in fact, the third requirement that is quite important in a direct determination of the propagator. We note that this last condition is missing at zero temperature which makes it difficult to determine the complete propagator (independent of the problem of divergence). When the theory is divergence free (so that it does not need a regularization at zero temperature), this propagator will be the exact fermion propagator and would lead to the complete effective action including the correct zero temperature part. On the other hand, if the theory needs to be regularized at zero temperature, this propagator will not yield the correct zero temperature effective action, but the finite temperature part of the effective action, which does not need to be regularized, will be determined correctly. We now illustrate the method with two examples.

Let us start with the $0+1$ dimensional QED described by the Lagrangian

$$L = \overline{\psi}(i\partial_t - m - eA(t))\psi, \tag{4}$$

where the fermion mass can be thought of as a chemical potential and in $0+1$ dimension, the gauge potential has only a single component. This is a simple model which has been studied exhaustively [9, 10] in connection with large gauge invariance [11] at finite temperature, but it is quite useful in clarifying what is involved in our proposal before we generalize to higher dimensions. As we noted earlier, we work with the closed time path formalism where the path in the complex time plane has the form shown in Fig. 1. In the closed time path formalism (in any real time formalism) [4], the degrees of freedom need to be doubled and we denote the background fields on the $\pm$ branches as $A_{\pm}(t)$ respectively. Since t is the only coordinate on which fields depend in this theory, there is no need for a mixed space propagator and we note that the complete fermion propagator of the theory (ordered along the contour in Fig. 1 satisfies the equation

$$\begin{aligned}(i\partial_t - m - eA_c(t))S_c(t,t') &= \delta_c(t-t'),\\ S_c(t,t')(i\overleftarrow{\partial}_{t'} + m + eA_c(t')) &= -\delta_c(t-t'),\end{aligned} \tag{5}$$

where the subscript "c" characterizes a function on the contour and the derivative $\partial_{t'}$ in the second equation acts to the left. On the contour, the step function is defined naturally as [4]

$$\theta_c(t-t') = \begin{cases} \theta(t-t') & \text{if both } t,t' \in C_+,\\ \theta(t'-t) & \text{if both } t,t' \in C_- \text{or } \in C_\perp,\\ 1 & \text{if } t\in C_- \text{and } t'\in C_+,\\ 0 & \text{if } t\in C_+ \text{and } t'\in C_-. \end{cases} \tag{6}$$

and we have $\delta_c(t-t') = \partial_t\theta_c(t-t')$.

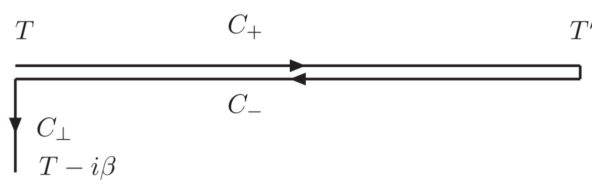

FIG. 1: The closed time path contour in the complex t plane. Here $T\to-\infty$, while $T'\to\infty$ while β denotes the inverse temperature (in units of the Boltzmann constant k). The two branches labelled by $\pm$ lead to the doubling of the degrees of freedom while the final branch along the imaginary axis decouples from any physical amplitude.

The equation (5) can be solved exactly subject to the three requirements leading to the contour Green's function of the form

$$\begin{aligned} S_c(t,t') &= e^{-imt - ie\int \mathrm{d}\bar{t}\,\theta_c(t-\bar{t})A_c(\bar{t})}\\ &\quad\times\left(\theta_c(t-t') - n_{\rm F}(m + \frac{ie}{\beta}(a_+ - a_-))\right)\\ &\quad\times e^{imt' + ie\int \mathrm{d}\bar{t}\,\theta_c(t'-\bar{t})A_c(\bar{t})}, \end{aligned} \tag{7}$$

where $n_{\rm F}$ denotes the Fermi distribution function with β corresponding to the inverse temperature and we have identified

$$a_{\pm} = \int_{-\infty}^{\infty} \mathrm{d}t\, A_{\pm}(t). \tag{8}$$

Although the phases in (7) can be combined to write them in a simpler form in this case, we have chosen to write it in this suggestive form which will generalize naturally to higher dimensions where the propagator will carry spinor indices. When t, t' are restricted to the appropriate branches of the contour, this determines the components of the full 2×2 matrix propagator of the theory. Furthermore, it can be checked that this propagator satisfies the Lippmann-Schwinger equation (the perturbation expansion) [12] for the propagator. However, we do not go into the details of these which will be published elsewhere.

The $0+1$ dimensional theory is free from divergences and, therefore, this represents the complete fermion propagator of the theory in the presence of a background gauge field. We can now take the coincident limit ($t = t'$) and integrate (2) to obtain the normalized effective action of the theory which has the form

$$\Gamma_{\rm eff}[a_+, a_-] = -i \ln \left[\cos \frac{e(a_+ - a_-)}{2} + i \tanh \frac{\beta m}{2} \sin \frac{e(a_+ - a_-)}{2} \right]. \quad (9)$$

This is the complete effective action of the theory which reduces to the well studied action [9, 10] on C_+ when we set $a_- = 0$. However, being the complete effective action, the action in (9) contains all the information about retarded, advanced and other amplitudes as well. For example, let us note from (9) that since $\Gamma_{\rm eff} = \Gamma_{\rm eff}[a_+ - a_-]$, the retarded n-point amplitude can be shown to vanish (for a definition of retarded amplitudes, see [13]), namely,

$$\begin{aligned} \Gamma_R^{(n)} &= \sum_{m=0}^{n-1} \left({}^{n-1}C_m \frac{\mathrm{d}^{n-1-m}}{\mathrm{d}a_+^{n-1-m}} \frac{\mathrm{d}^m}{\mathrm{d}a_-^m} \right) \frac{\mathrm{d}\Gamma_{\rm eff}[a_+ - a_-]}{\mathrm{d}a_+} \Bigg| \\ &= (1-1)^{n-1} \frac{\mathrm{d}^n \Gamma_{\rm eff}[a_+ - a_-]}{\mathrm{d}a_+^n} \Bigg| = 0, \quad n \geq 2, \end{aligned} \quad (10)$$

where the restriction stands for setting all the background fields to zero. Therefore, all the retarded (similarly, advanced) amplitudes vanish in this theory. (The one point amplitude is by definition a Feynman amplitude.)

With this brief description of the derivation of the complete effective action in the 0+1 dimensional theory, let us next consider the fermion sector of the Schwinger model [8] or massless QED in $1+1$ dimensions described by the Lagrangian density

$$\mathcal{L} = \overline{\psi}(t, x) \gamma^\mu (i\partial_\mu - eA_\mu(t, x)) \psi(t, x). \quad (11)$$

The effective action for this model has been studied perturbatively at finite temperature [14] even in the presence of a chemical potential [15]. Here we will derive the closed form of the finite temperature effective action following our method. We note here that the two point function in the Schwinger model needs to be regularized at zero temperature and, consequently, the zero temperature part following from our propagator will not coincide with the zero temperature effective action. However, our interest is in the finite temperature part of the effective action which is free from ultraviolet divergences. For completeness we note that a simple point-splitting regularization of the fermion propagator is sufficient to regularize the theory and can be carried out even in our method. However, we will not do this since our main focus is in the finite temperature part of the effective action.

The theory (11) is best studied in the natural basis of right handed and left handed fermion fields (although everything that we say can be carried out covariantly as well as in the presence of a chemical potential). Defining [16]

$$\begin{aligned} \psi_{\rm R} &= \frac{1}{2}(\mathbb{1} + \gamma_5)\psi, \quad \psi_{\rm L} = \frac{1}{2}(\mathbb{1} - \gamma_5)\psi, \\ x^\pm &= \frac{x^0 \pm x^1}{2}, \quad p_\pm = p_0 \pm p_1, \quad \partial_\pm = \partial_0 \pm \partial_1, \\ A_\pm &= A_0 \pm A_1, \end{aligned} \quad (12)$$

the Lagrangian density (11) naturally decouples into two sectors described by the Lagrangian density

$$\mathcal{L} = \psi_{\rm R}^\dagger (i\partial_+ - eA_+)\psi_{\rm R} + \psi_{\rm L}^\dagger (i\partial_- - eA_-)\psi_{\rm L}, \quad (13)$$

where $\psi_{\rm R}, \psi_{\rm L}$ here denote component spinor fields (no spinor index left any more). While the zero temperature regularization (anomaly) mixes the two sectors through the two point function, at finite temperature we do not have divergences and we do not expect the two sectors to mix. Therefore, we can study the finite temperature effective action in the two sectors separately.

Let us consider the theory only in the sector of the right handed fermions in (13). This is very much like the $0+1$ dimensional theory. However, there is one essential difference that makes the derivation much more difficult, namely, the field variables depend on two coordinates (t, x) or equivalently on (x^+, x^-). We would like to emphasize here that although we use the light-cone coordinates for simplicity, the theory is still quantized on the equal-time surface and the propagator is defined through the time ordered Green's function (namely, we do not use the statistical mechanics of the light-front [17]). As we mentioned earlier, the finite temperature derivations become a lot simpler in the mixed space. Thus, Fourier transforming the x^- coordinate, the action for the right handed fermions takes the form (the conjugate variable to x^- should be p_-, which we write as p for simplicity)

$$S_{\rm R} = 2 \int \mathrm{d}x^+ \frac{\mathrm{d}p}{2\pi} \psi_{\rm R}^\dagger(x^+, -p) \left[i\partial_+ \psi_{\rm R}(x^+, p) - e \int \frac{\mathrm{d}k}{2\pi} A_+(x^+, p-k) \psi_{\rm R}(x^+, k) \right]. \quad (14)$$

As a result, we recognize that the equation for the Green's function will involve a convolution. It is best described by introducing the operator notation for the Green's function as well as the gauge potential

$$\begin{aligned} S(x^+, x'^+; p) &= \int \frac{\mathrm{d}k}{2\pi} \langle p | \hat{S}(X^+, x'^+) | k \rangle, \\ A_+(x^+, p-k) &= \langle p | \hat{A}_+(x^+) | k \rangle, \end{aligned} \quad (15)$$

so that the equation for the Green's function on the contour takes the (operator) form (see also (5))

$$(i\partial_+ - e\hat{A}_c(x^+))\hat{S}_c(x^+,x'^+) = \frac{i}{2}\,\delta_c(x^+ - x'^+),$$
$$\hat{S}_c(x^+,x'^+)(i\partial'_+ + e\hat{A}_c(x'^+)) = -\frac{i}{2}\,\delta_c(x^+ - x'^+).\quad(16)$$

The solution to (16) satisfying the Ward identity as well as the appropriate antiperiodicity condition can be determined to have the form

$$\begin{aligned}\hat{S}_c(x^+,x'^+) \;&=\; e^{-ie\int \mathrm{d}\bar{x}^+\,\theta_c(x^+-\bar{x}^+)\hat{A}_c(\bar{x}^+)}\\ &\times\left(\frac{-i}{4}\right)\left(\mathrm{sgn}(x^+ - x'^+) + \hat{\overline{\mathcal{O}}}_+\right)\\ &\times e^{ie\int \mathrm{d}\bar{x}^+\,\theta_c(x'^+-\bar{x}^+)\hat{A}_c(\bar{x}^+)},\end{aligned}\quad(17)$$

where

$$\begin{aligned}\hat{\overline{\mathcal{O}}}_+ \;&=\; 1 - 2(\hat{\mathcal{O}}_+^2 + 1)^{-1},\\ \hat{\mathcal{O}}_+^2 \;&=\; e^{\frac{ie(\hat{a}_{+(+)}-\hat{a}_{+(-)})}{2}}\, e^{\frac{\beta K}{2}}\, e^{\frac{ie(\hat{a}_{+(+)}-\hat{a}_{+(-)})}{2}},\end{aligned}\quad(18)$$

with K denoting the momentum operator and (($\pm$) with the parenthesis denote the thermal indices while $+$ without the parenthesis represents the light-cone component of the background field)

$$\hat{a}_{+(\pm)} = \int_{-\infty}^{\infty} \mathrm{d}x^+\,\hat{A}_{+(\pm)}(x^+).\quad(19)$$

It can be checked that this complete propagator satisfies the Lippmannn-Schwinger equation. Furthermore, setting $x^+ = x'^+$ and using (3) we can determine the normalized effective action in the right handed sector to be

$$\begin{aligned}\Gamma_{\mathrm{R,\,eff}} \;&=\; \int \mathrm{d}x^+ \int \frac{\mathrm{d}p\mathrm{d}k}{(2\pi)^2}\\ &\times\langle p|\ln\frac{(\hat{\mathcal{O}}_+ + \hat{\mathcal{O}}_+^{-1})}{2} - \ln\cosh\frac{\beta K}{2}|k\rangle.\end{aligned}\quad(20)$$

The thermal part of this effective action has the right (delta function) structure that had already been observed in the perturbative calculation in the right handed sector [14] which is a consequence of the Ward identity in the theory. In fact, the expansion of this effective action agrees order by order with the perturbative result. The effective can similarly be determined for the left handed sector to be (here p, k should be understood as the conjugate variables to x^+ which should be written as p_+, k_+)

$$\begin{aligned}\Gamma_{\mathrm{L,\,eff}} \;&=\; \int \mathrm{d}x^- \int \frac{\mathrm{d}p\mathrm{d}k}{(2\pi)^2}\\ &\times\langle p|\ln\frac{(\hat{\mathcal{O}}_- + \hat{\mathcal{O}}_-^{-1})}{2} - \ln\cosh\frac{\beta K}{2}|k\rangle,\end{aligned}\quad(21)$$

with $\hat{\mathcal{O}}_-$ obtained from $\hat{\mathcal{O}}_+$ in (18) with

$$\hat{a}_{+(\pm)}(x^+) \to \hat{a}_{-(\pm)}(x^-).\quad(22)$$

Once again, this effective action has the right (delta function) structure as in the perturbative calculation and agrees with the perturbative result order by order. The finite temperature effective action for a $1+1$ dimensional fermion interacting with a gauge background can, therefore, be obtained from

$$\Gamma_{\mathrm{eff}} = \Gamma_{\mathrm{R,eff}} + \Gamma_{\mathrm{L,eff}}\quad(23)$$

which, of course, leads to the perturbative result order by order, but the effective action being a functional of $(\hat{a}_{\pm(+)} - \hat{a}_{\pm(-)})$, it can be checked as in (10) that all the retarded amplitudes in this theory vanish. This should be contrasted with the fact that this is only known explicitly up to the 3-point function in perturbation theory (although believed to be true for all amplitudes).

In summary, we have proposed an alternative method for determining the finite temperature effective action for fermions interacting with a background field by determining the complete fermion propagator directly by using the anti-periodicity appropriate at finite temperature in the closed time path formalism. We have illustrated how our proposal works with the examples of the $0+1$ dimensional QED as well as in the case of the Schwinger model. We have only sketched the derivation without too many details in this paper. The details of the calculations as well as other aspects of this analysis will be published separately.

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[1] J. Schwinger, Phys. Rev. **82**, 664 (1951).
[2] W. Dittrich, Phys. Rev. **D19**, 23 (1978); P. H. Cox, and W. S. Hellman, Ann. Phys. **154**, 211 (1984); M. Loewe and J. C. Rojas, Phys. Rev. **D46**, 2689 (1992); P. Elmfors, D. Persson and B.-S. Skagerstam, Phys. Rev. Lett. **71**, 480 (1993); P. Elmfors and B.-S. Skagerstam, Phys. Lett. **B348**, 141 (1995); A. Das and J. Frenkel, Phys. Rev. **D75**, 025021 (2007).
[3] J. Kapusta, *Finite Temperature Field Theory*, Cambridge University Press, Cambridge, England (1989); M. Le Bellac, *Thermal Field Theory*, Cambridge University Press, Cambridge, England (1996).
[4] A. Das, *Finite Temperature Field Theory*, World Scientific, Singapore (1997).
[5] J. Schwinger, *Lecture Notes of Brandeis Summer Institute in Theoretical Physics* (1960); J. Schwinger, J. Math.

Phys. **2**, 407 (1961); P. M. Bakshi and K. T. Mahanthappa, J. Math. Phys. **4**, 1 (1963); L. V. Keldysh, Sov. Phys. JETP **20**, 1018 (1965).

[6] T. Matsubara, Prog. Theor. Phys. **14**, 351 (1954).

[7] F. T. Brandt, A. Das, O. Espinosa, J. Frenkel and S. Perez, Phys. Rev. **D72**, 085006 (2005); *ibid* **D73**, 065010 (2006); *ibid* **D73**, 067702 (2006).

[8] J. Schwinger, Phys. Rev. **128**, 2425 (1962).

[9] G. Dunne, K. Lee and C. Lu, Phys. Rev. Lett. **78**, 3434 (1997)

[10] A. Das and G. Dunne, Phys. Rev. **D57**, 5023 (1998); J. Barcelos-Neto and A. Das, Phys. Rev. **D58**, 085022 (1998).

[11] K. S. Babu, A. Das and P. Panigrahi, Phys. Rev. **D36**, 3725 (1987).

[12] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

[13] F. T. Brandt, A. Das, J. Frenkel and A. J. da Silva, Phys. Rev. **D59**, 065004 (1999); F. T. Brandt, A. Das and J. Frenkel, Phys. Rev. **D60**, 105008 (1999).

[14] A. Das and A. J. da Silva, Phys. Rev. **D59**, 105011 (1999).

[15] S. Maciel and S. Perez, Phys. Rev. **D78**, 065005 (2008).

[16] For quantization of massless fermion fields, see, for example, *Lectures on Quantum Field Theory*, A. Das, World Scientific Publishing, Singapore (2008).

[17] V. S. Alves, A. Das and S. Perez, Phys. Rev. **D66**, 125008 (2002).