Today in Astronomy 102: general relativity and the prediction of the existence of black holes

- Schwarzschild solves Einstein's equations, applied to stars, and finds black holes in the results.
- Singularities in physics: why nobody worried about Schwarzschild's black holes for a while.

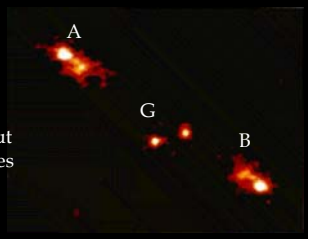


Image: gravitational lens in galaxy cluster AC114 (Smail et al. 1995). Note that the "quasars" A and B are mirror images of each other; they are both images of a single quasar lying far behind the cluster to which galaxies G belong.

Karl Schwarzschild's Work

In 1916 Schwarzschild read Einstein's paper on general relativity. He was interested in the physics of stars, and had a lot of spare time between battles on the Russian front, so he solved Einstein's field equation for the region outside a massive spherical object.

His solution had many interesting features, including

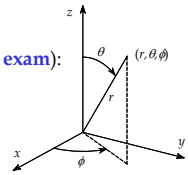
- prediction of space warping in strong gravity, and invention of embedding diagrams to visualize it.
- demonstration of gravitational time dilation, just as Einstein had pictured it.
- prediction of black holes, though this was not recognized at the time.

Karl Schwarzschild

Hieroglyphics: part of Schwarzschild's solution

In case you're interested (i.e. **not on the exam**):

- In spherical coordinates (at right), Schwarzschild's absolute interval Δs, between two events that occur very close to a circle (circumference C) around the star, is



$$\Delta s = \sqrt{\frac{\Delta r^2}{1-\frac{4\pi GM}{Cc^2}} + \frac{C^2\Delta\theta^2}{4\pi^2} + \frac{C^2\sin^2\theta\Delta\phi^2}{4\pi^2} - c^2\left(1-\frac{4\pi GM}{Cc^2}\right)\Delta t^2}$$

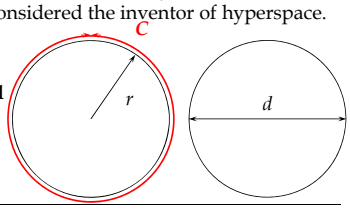
Note that some of the terms blow up (zero in the denominator), or are zero, if $C = 4\pi GM/c^2$.

- This formula is used in calculations on pp. 6-7.

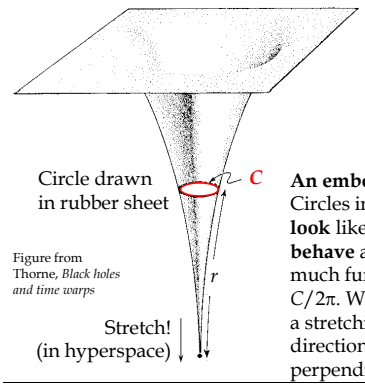
Embedding diagrams, and why they're useful

Embedding ("rubber sheet") diagrams provide an **analogy** by which one can envision how four-dimensional spacetime can be warped, by introduction of additional, imaginary dimensions. These additional dimensions have been introduced here before, under the name **hyperspace.** Schwarzschild can be considered the inventor of hyperspace.

First, consider circles in flat **two-dimensional** space:
$C = 2\pi r = \pi d$



Embedding diagrams, and why they're useful (continued)



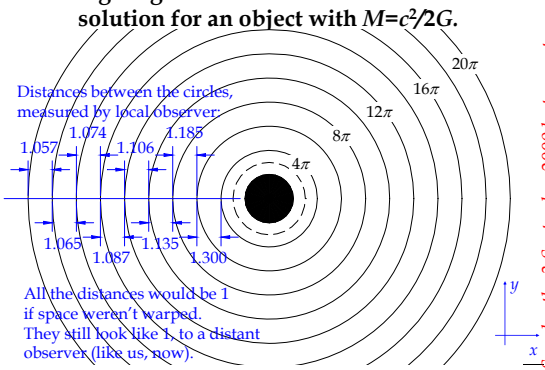
Circle drawn in rubber sheet

Figure from Thorne, *Black holes and time warps*

Stretch! (in hyperspace)

An embedding diagram. Circles in curved space still **look** like 2-D figures, but **behave** as if their centers are much further away than $C/2\pi$. We can picture this as a stretching of space in a direction (in hyperspace!) perpendicular to the circle.

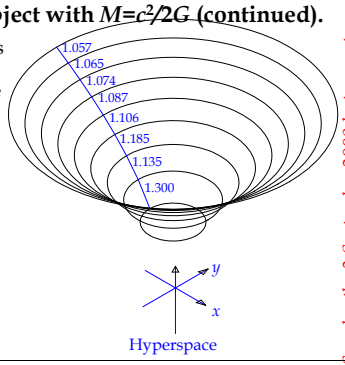
Embedding diagram calculated from Schwarzschild's solution for an object with $M=c^2/2G$.



See also the 3 September 2009 lecture notes.

Embedding diagram calculated from Schwarzschild's solution for an object with $M=c^2/2G$ (continued).

To connect these circles with segments of these "too long" lengths, one can consider them to be offset from one another along some **imaginary** dimension that is **perpendicular to x and y but is not z**. (If it were z, the circles wouldn't appear to lie in a plane!). Such additional dimensions comprise hyperspace.



See also the 3 September 2009 lecture notes.

Embedding diagrams, and why they're useful (continued)

$C = \pi d$ if spacetime is flat (i.e. if gravity is weak)

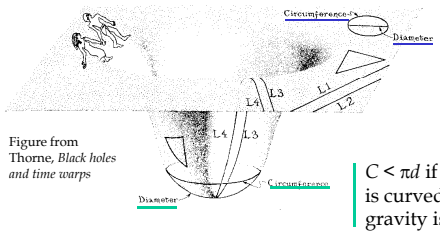


Figure from Thorne, *Black holes and time warps*

$C < \pi d$ if spacetime is curved (i.e. if gravity is strong)

Schwarzschild's view of the equatorial plane of a star

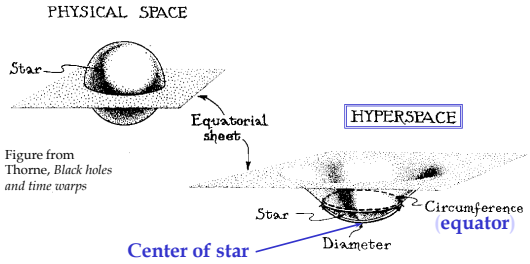


Figure from Thorne, *Black holes and time warps*

Mid-lecture break.

Exam #1 takes place this Thursday, 1 October 2009.

- As previously announced, a practice exam is available on WeBWorK: follow the link on the AST 102 home page.
- The exam is given on WeBWorK, and works like the practice exam, except…
- you will have 75 minutes to complete the test once you start. You can choose any 75-minute window between noon and 6PM.

Third panel of *The Garden of Earthly Delights*, Hieronymus Bosch (Museo del Prado).

PRSs near the horizon

You are in an orbit around a black hole, with circumference **10 times that of the BH's horizon**, and you can measure distances with 10% accuracy. By switching to different orbits very near by, and measuring circumferences and distances between orbits, can you reveal the space-warping effects of the BH?

A. Yes; the BH warps space severely here. B. Yes, but just barely.
C. No; accuracy insufficient. D. No; the warp is essentially zero here.

PRSs return to the horizon

You change to an orbit around a **much more massive black hole**, again with circumference 10 times that of the BH's horizon, and again you can measure distances with 10% accuracy. By switching to different orbits very near by, and measuring circumferences and distances between orbits, can you reveal the space-warping effects of the BH?

A. Yes; the BH warps space severely here. B. Yes, but just barely.
C. No; accuracy insufficient. D. No; the warp is essentially zero here.

Children of the return of the PRSs to the horizon

You change to an orbit with circumference 3 times that of the BH's horizon, and again you can measure distances with 10% accuracy. By switching to different orbits very near by, and measuring circumferences and distances between orbits, can you reveal the space-warping effects of the BH?

A. Yes; the BH warps space severely here. B. Yes, but just barely.

C. No; accuracy insufficient. D. No; the warp is essentially zero here.

Schwarzschild's solution to the Einstein field equations

Results: the curvature of spacetime outside a massive star.

- For a given, fixed star mass M, he found how space and time are curved if the star is made smaller in size.
- **Singularity**: if the star is made smaller than a certain critical size,

$$C_S = \frac{4\pi GM}{c^2},$$ Schwarzschild circumference

We will be using this formula, but not on Exam #1.

the gravitational redshift of light – time dilation, remember – predicted by his solution is **infinite**!
(Here, $G = 6.67\times10^{-8}$ cm^3/(gm sec^2) is Newton's gravitational constant, and $c = 2.9979\times10^{10}$ cm/sec is, as usual, the speed of light (and $\pi = 3.14159....$).)

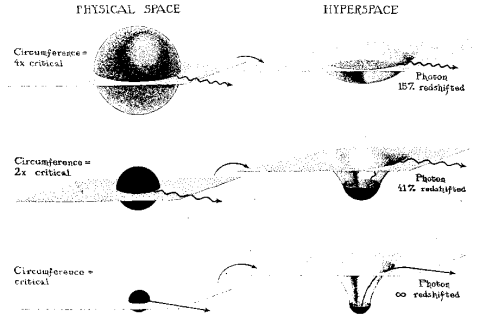


Figure from Thorne, *Black holes and time warps*

Greatest hits of the children of the return of the PRS

Schwarzschild found that as the size of a white star was reduced and began to approach the size of the singularity, it would begin to appear to an observer as

A. spherical and red. B. funnel-shaped and red.

C. spherical and white. D. funnel-shaped and white.

Implications of "Schwarzschild's singularity"

- If a star is made too small in circumference for a given mass, nothing can escape from it, **not even light**.
  - This would be a black hole, and the critical size is the size of the black hole's horizon.
- This is similar to an 18th century idea: "dark stars" (John Michell and Pierre Laplace, independently); if light were subject to gravitational force, there could be stars from which light could not escape.
  - The critical size of a star with Schwarzschild's singularity turns out to be the same as Michell and Laplace determined (from classical physics!) for their "dark star".

Singularities in physics, math and astronomy

A formula is called **singular** if, when one puts the numbers into it in a calculation, the result is infinity, or is not well defined. The particular combination of numbers is called the **singularity**.

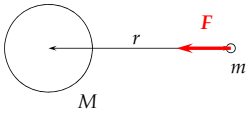
Singularities often arise in the formulas of physics and astronomy. They usually indicate either:

- **invalid approximations** -- not all of the necessary physical laws have been accounted for in the formula (no big deal), or
- that the singularity is **not realizable** (also no big deal), or
- that a mathematical **error** was made in obtaining the formula (just plain wrong).

They are hardly ever real: infinity is hard to come by!

Singularities in physics, math and astronomy (continued)

Example of a classical physics law with a singularity: Newton's law of gravitation.



$$F = \frac{GMm}{r^2}$$

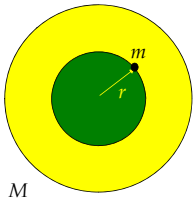
- r is the distance between the centers of the two spherical masses. A spherical mass exerts force as if its mass is concentrated at its center.
- Clearly, if r were zero, the force would be infinite!

This formula will not appear on homework or exams. It is used only because it is a good example of a singularity.

Singularities in physics, math and astronomy (continued)

This singularity is **not realized**, however, because:

- **the mass M really isn't concentrated at a point.**
- a spherical shell of matter does not exert a net gravitational force on a mass inside it.



Consider mass m inside mass M: outer (yellow) matter's forces on m cancel out, and only inner (green) exerts a force. As m gets closer to the center ($r \to 0$), the force gets smaller, not larger.

No singularity!

Reaction to the Schwarzschild singularity

- Schwarzschild's solution to the Einstein field equation was demonstrated to be correct - the singularity is not the result of a math error.
- Thus most physicists and astronomers assumed that the singularity would not be physically realizable (just like the singularity in Newton's law of gravitation) or that accounting for other physical effects would remove it.
- Einstein (1939) eventually tried to prove this in a general-relativistic calculation of stable (non-collapsing or exploding) stars of size equal to the Schwarzschild circumference.
- He found that this would require **infinite gas pressure**, or **particle speed greater than the speed of light**, both of which are impossible.

Reaction to the Schwarzschild singularity (continued)

- Einstein's results show that a stable object with a singularity cannot exist.
- From this he concluded (**incorrectly**) that this meant the singularity could not exist in nature.
- Einstein's calculation was correct, but the correct inference from the result is that **gas pressure cannot support the weight of stars similar in size to the Schwarzschild circumference**.
- If nothing stronger than gas pressure holds them up, such stars will collapse to form black holes -- the singularity is **real**. (!!?)
  - Does physics provide us with any such pressure? Find out next Tuesday.