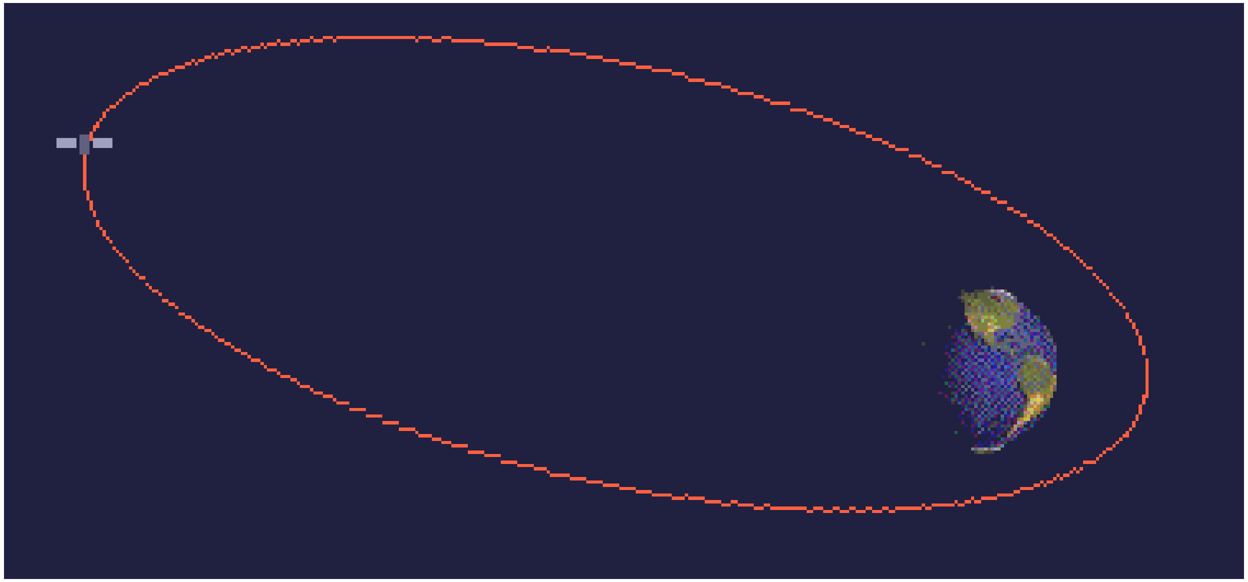
Today in Astronomy 111: Newton's and Kepler's laws, planetary orbits

Before we move on from Mars to the asteroids, we need to learn a bit about celestial mechanics, starting with:

- ❑ Uniform circular motion
- ❑ Center of mass
- ❑ Elliptical orbits and their consistency with Newtonian mechanics
- ❑ Kepler's laws
- ❑ (Pre-)validation of Newtonian dynamics by Kepler and Tycho



Review: Newton's laws

Newton's laws of motion:

- ❑ Newton's first law: It takes force to change a body's velocity, either in magnitude or direction.
- ❑ Newton's second law: The rate at which a body changes its velocity (i.e. accelerates) is proportional to the force:

$$\boldsymbol{F} = m\boldsymbol{a}$$

Note that force ***F*** and acceleration ***a*** are vectors.

- ❑ Newton's third law: Every force is opposed by an equal and opposite force in reaction.

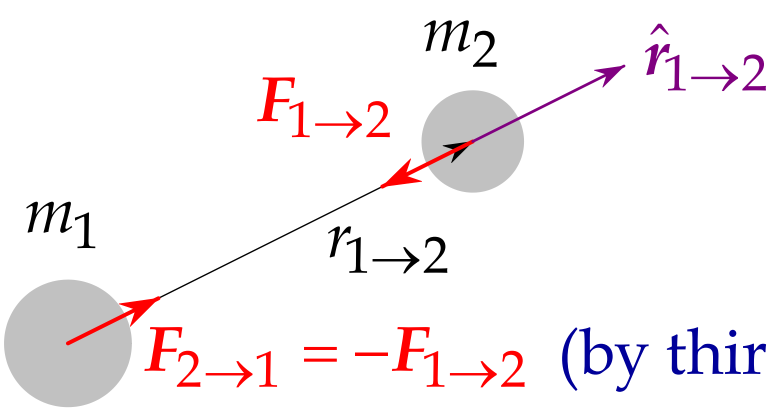
Review: Newton's laws (continued)

Newton's law of gravity, force and potential energy:

$$\boldsymbol{F}_{1\to2} = -\frac{Gm_1m_2}{r_{1\to2}^2}\hat{\boldsymbol{r}}_{1\to2} \quad \Leftrightarrow \quad U_{1,2} = -\frac{Gm_1m_2}{r_{1\to2}}$$

where $G = (6.674215 \pm 0.000092)\times10^{-8}\ \mathrm{cm^3\ gm^{-1}\ s^{-2}}$

$\times10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}}$



$\boldsymbol{F}_{2\to1} = -\boldsymbol{F}_{1\to2}$ (by third law)

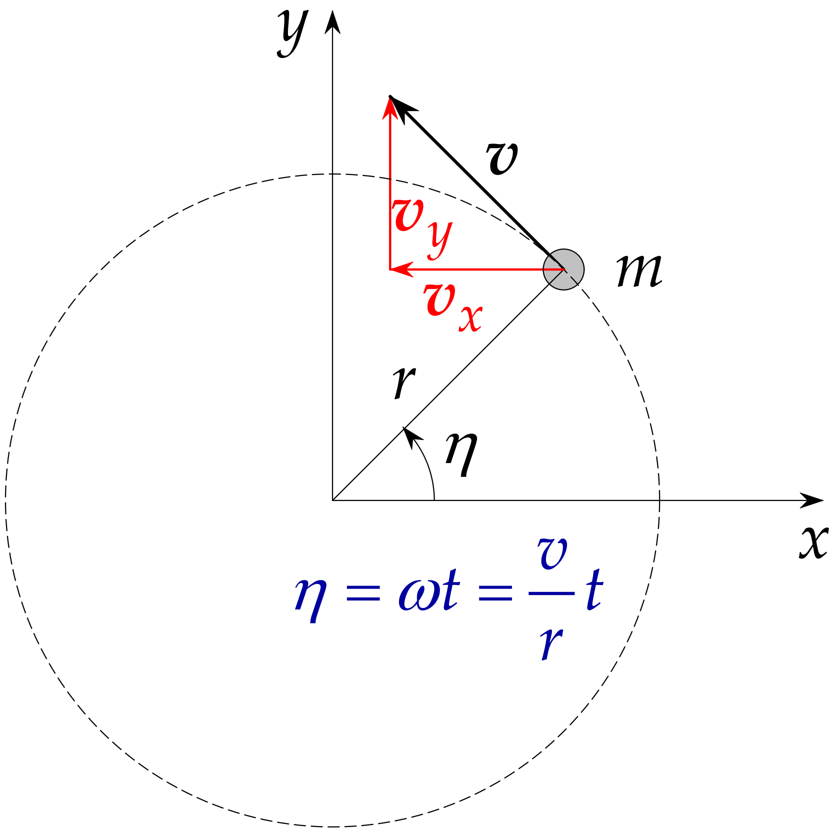
Note: spherical bodies act gravitationally on objects outside them as if all their mass is concentrated at their centers.

Uniform circular motion

Suppose an object (mass m) moves in a circle at constant speed v. (Speed = magnitude of velocity.)

- Then it accelerates, because its velocity is constantly changing direction, but the acceleration is always perpendicular to the velocity, because the speed doesn't change. **What is the acceleration?**

  Break velocity into Cartesian components:



Uniform circular motion (continued)

- The angle v makes with the x axis is $\eta + \pi/2$, so

$$v_x = v\cos\left(\omega t + \pi/2\right) = -v\sin\omega t$$

$$v_y = v\sin\left(\omega t + \pi/2\right) = v\cos\omega t$$

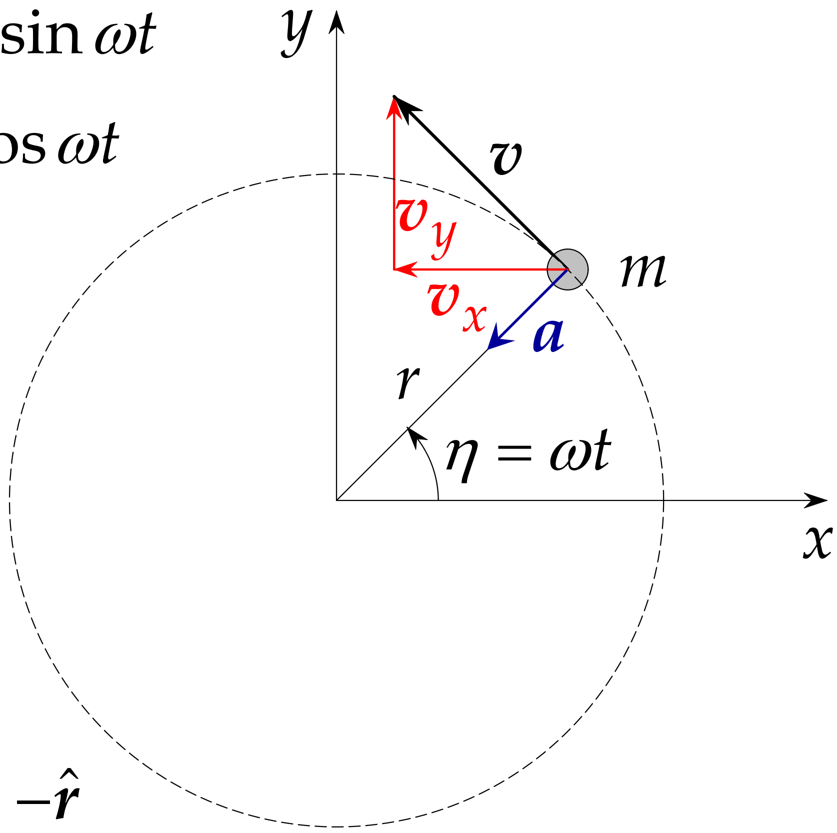
Thus $$a_x = \frac{dv_x}{dt} = -\omega v\cos\omega t$$

$$a_y = \frac{dv_y}{dt} = -\omega v\sin\omega t$$

Centripetal acceleration

$$a = \sqrt{a_x^2 + a_y^2} = \omega v = \frac{v^2}{r}$$

$$\hat{\boldsymbol{a}} = -\hat{\boldsymbol{x}}\cos\omega t - \hat{\boldsymbol{y}}\sin\omega t = -\hat{\boldsymbol{r}}$$



Uniform circular motion (continued)

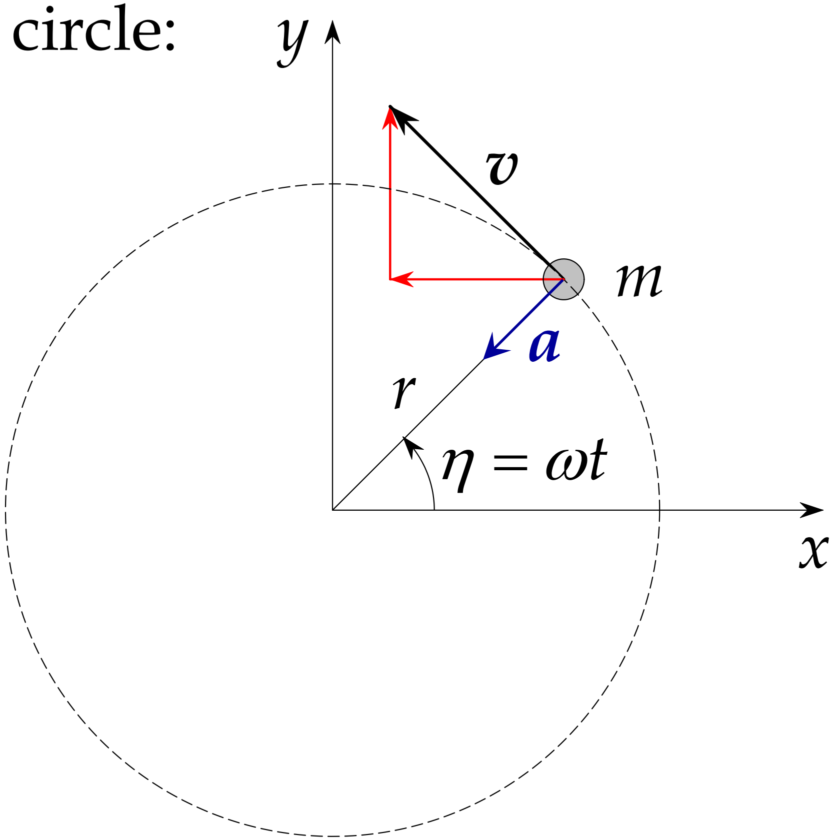
That is, a body in uniform circular motion at speed v and radius r accelerates with constant magnitude $a = v^2/r$, directed toward the center of the circle:

$$\boldsymbol{a} = -\frac{v^2}{r}\hat{\boldsymbol{r}} \quad .$$

- So, unsurprisingly, this would require a force in the $-\hat{\boldsymbol{r}}$ direction.

Note that because v and thus ω are constant in circular orbit, there is a simple relation with period P:

$$P = 2\pi r/v = 2\pi/\omega \quad .$$



Uniform circular motion and gravity

Suppose a large mass M lies a distance r from a mass m. At what speed will the small mass orbit – in a circle – about the large mass?

$$\boldsymbol{F} = -\frac{GMm}{r^2}\hat{\boldsymbol{r}} = m\boldsymbol{a} = -m\frac{v^2}{r}\hat{\boldsymbol{r}}$$

$$\boxed{v = \sqrt{GM/r}}$$

- What is the corresponding total energy, and the angular momentum, of the small mass relative to the center of its orbit?

$$E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}$$

$$\boldsymbol{L} = \boldsymbol{r} \times m\boldsymbol{v} = \hat{\boldsymbol{z}}rmv\sin\frac{\pi}{2} = \hat{\boldsymbol{z}}m\sqrt{GMr}$$

Center of mass

What if the masses are not very different? Then both move in an orbit about their center of mass. The **center of mass** is defined according to momentum conservation. In a frame of reference in which the bodies have zero momentum, the position of the center of mass is constant in time:

$$(m_1 + m_2)\frac{d\mathbf{r}_{CM}}{dt} = m_1\frac{d\mathbf{r}_1}{dt} + m_2\frac{d\mathbf{r}_2}{dt} = 0$$

$$(m_1 + m_2)\mathbf{r}_{CM} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \text{constant}$$

$$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

(constant →) 0, for the most convenient coordinate – origin choice.

Center of mass (continued)

Suppose the two masses are separated by displacement $\boldsymbol{r}$, and we place the center of mass at the origin of coordinates:

$$\frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2} = 0 \quad , \quad \boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$$

$$\frac{m_1\boldsymbol{r}_1 + m_2\left(\boldsymbol{r} + \boldsymbol{r}_1\right)}{m_1 + m_2} = 0 \quad \Rightarrow \quad \boldsymbol{r}_1 = -\frac{m_2}{m_1 + m_2}\boldsymbol{r}$$

Similarly,

$$\boldsymbol{r}_2 = \frac{m_1}{m_1 + m_2}\boldsymbol{r}$$

Usually one also defines the **reduced mass**:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Not to be confused with the μ used briefly in the textbook.

Center of mass (continued)

Then $$\mathbf{r}_1 = -\frac{\mu}{m_1}\mathbf{r} \quad , \quad \mathbf{r}_2 = \frac{\mu}{m_2}\mathbf{r} \quad .$$

Thus $$\mathbf{v}_1 = -\frac{\mu}{m_1}\frac{d\mathbf{r}}{dt} = -\frac{\mu}{m_1}\mathbf{v} \quad , \quad \mathbf{v}_2 = \frac{\mu}{m_2}\mathbf{v} \quad ,$$

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{G m_1 m_2}{\left|\mathbf{r}_2 - \mathbf{r}_1\right|}$$

$$= \frac{1}{2}\mu v^2 - \frac{G\mu\left(m_1 + m_2\right)}{r}$$

Much like the large-mass case.

The motion of one of the two bodies, in a reference frame in which the other is at rest, is the same as the motion of the reduced mass.

Center of mass (continued)

Example 1. *Describe the orbits of two masses, m_1 and m_2, about their common center of mass, if their separation is r.*

In a coordinate system centered on m_1, the force on m_2 appears as

$$\boldsymbol{F}_2 = -\frac{Gm_1 m_2}{r^2}\hat{\boldsymbol{r}} = m_2 \boldsymbol{a}_2' = -m_2 \frac{v_2'^2}{r}\hat{\boldsymbol{r}} \quad \Rightarrow \quad v_2' = \sqrt{\frac{Gm_1}{r}}$$

that is, circular motion about m_1. But in a coordinate system with its origin at the center of mass,

$$\boldsymbol{r}_1 = -\frac{\mu}{m_1}\boldsymbol{r} \quad , \quad \boldsymbol{r}_2 = \frac{\mu}{m_2}\boldsymbol{r} \quad ,$$

$$\boldsymbol{v}_1 = -\frac{\mu}{m_1}\frac{d\boldsymbol{r}}{dt} = -\frac{\mu}{m_1}\boldsymbol{v}_2' = -\frac{\mu}{m_1}\sqrt{\frac{Gm_1}{r}} \quad , \quad \boldsymbol{v}_2 = \frac{\mu}{m_2}\sqrt{\frac{Gm_1}{r}} \quad ,$$

Center of mass (continued)

so each mass travels in a circular orbit about the center of mass, one with radius $r_1 = -\mu \mathbf{r}/m_1$ and speed $\mathbf{v}_1 = -\frac{\mu}{m_1}\sqrt{\frac{Gm_1}{r}}$,

one with radius $r_2 = -\mu \mathbf{r}/m_2$ and speed $\mathbf{v}_2 = -\frac{\mu}{m_2}\sqrt{\frac{Gm_1}{r}}$,

- Note that if $m_1 = m_2 = m,$ the masses are equidistant from the center of mass, and orbit at the same speed:

$$r_{=} = \frac{m}{m+m} r = \frac{r}{2} \quad ,$$

$$\mathbf{v}_{=} = \frac{m}{m+m}\sqrt{\frac{Gm}{r}} = \frac{1}{2}\sqrt{\frac{Gm}{r}} \quad .$$

Escape speed

If an object is gravitationally bound to another one, with total energy E (a negative number), and an external agent adds a kinetic energy to the object equal to $-E$, then the object can (just barely) escape.

- **Example 2**: *escape from the surface of an* ***isolated*** *planet with mass M, radius R, initially at rest but given an impulse:*

$$E_i = K_i + U_i = -\frac{GMm}{R}$$

$$E_f = K_f + U_f = \frac{1}{2}mv_{esc}^2 - \frac{GMm}{R} = 0$$

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

$\sqrt{2}$ times the orbital speed at distance R.

Elliptical orbits

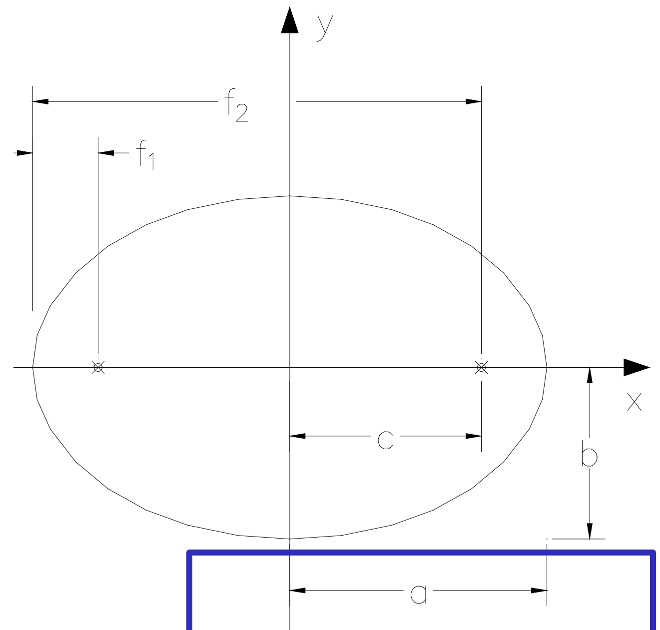
We reminded ourselves of the properties of ellipses a few days ago (lecture, 15 September 2011):

Equation $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Foci at $x=\pm c$: $$c=\sqrt{a^2-b^2}$$

Eccentricity $$\varepsilon=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}$$

Focal lengths (aphelion and perihelion distances) $$f=a\pm c$$ $$=a(1\pm\varepsilon)$$



Do not confuse semimajor axis with acceleration.

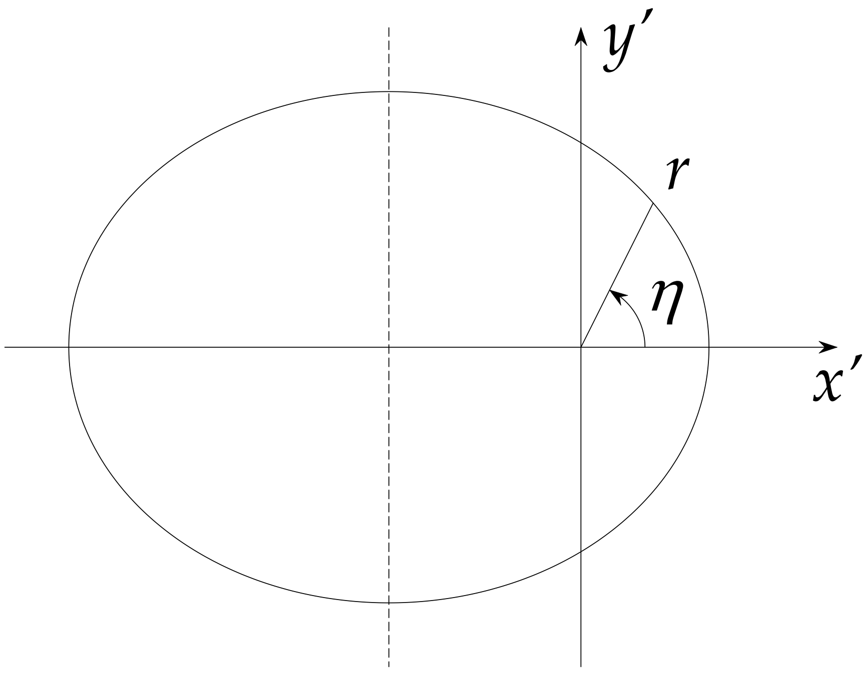
Elliptical orbits (continued)

It is a simple algebraic exercise to show that by substituting

$$x' = r\cos\eta \quad \text{and} \quad y' = r\sin\eta \quad \text{into} \quad \frac{(x'+c)^2}{a^2} + \frac{y^2}{b^2} = 1$$

that is, polar coordinates with origin on one focus and angle measured from the correspon-ding major axis, the equation for the ellipse becomes

$$r = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\eta} \quad .$$



The algebra we won't discuss, I

First eliminate b and c in favor of a and ε:

$$c = \varepsilon a$$

$$b^2 = a^2 - c^2 = a^2\left(1-\varepsilon^2\right)$$

Then the equation of the ellipse becomes

$$\frac{\left(x'+c\right)^2}{a^2} + \frac{y'^2}{b^2} = \frac{\left(x'+\varepsilon a\right)^2}{a^2} + \frac{y'^2}{a^2\left(1-\varepsilon^2\right)} = 1$$

$$\frac{r^2\cos^2\eta + \varepsilon^2 a^2 + 2r\varepsilon a\cos\eta}{a^2} + \frac{r^2\sin^2\eta}{a^2\left(1-\varepsilon^2\right)} = 1$$

The algebra we won't discuss, I (continued)

Multiply through:

$$\left(r^2\cos^2\eta+\varepsilon^2a^2+2r\varepsilon a\cos\eta\right)\left(1-\varepsilon^2\right)+r^2\sin^2\eta=a^2\left(1-\varepsilon^2\right)$$

$$r^2\left(1-\varepsilon^2\cos^2\eta\right)+r\left[2\varepsilon a\cos\eta\left(1-\varepsilon^2\right)\right]-a^2\left(1-\varepsilon^2\right)^2=0$$

This is a quadratic equation in r, with solution

$$r=\frac{1}{2\left(1-\varepsilon^2\cos^2\eta\right)}\left[-2\varepsilon a\cos\eta\left(1-\varepsilon^2\right)\right.$$

$$\left.\pm\sqrt{4\varepsilon^2a^2\cos^2\eta\left(1-\varepsilon^2\right)^2+4\left(1-\varepsilon^2\cos^2\eta\right)a^2\left(1-\varepsilon^2\right)^2}\right]$$

The algebra we won't discuss, I (continued)

The $b^2 - 4ac$ term turns out to simplify considerably:

$$r = \frac{1}{\left(1-\varepsilon^2\cos^2\eta\right)}\left(-\varepsilon a\cos\eta\left(1-\varepsilon^2\right) \pm a\left(1-\varepsilon^2\right)\right)$$

$$= a\left(1-\varepsilon^2\right)\frac{\pm 1-\varepsilon\cos\eta}{\left(1+\varepsilon\cos\eta\right)\left(1-\varepsilon\cos\eta\right)} = \mp\frac{a\left(1-\varepsilon^2\right)}{1\mp\varepsilon\cos\eta}$$

Normally r is considered to be a positive number, so we choose the lower sign:

$$r = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\eta}$$

Elliptical orbits (continued)

Suppose a very large mass M occupies the origin, and a small mass m is in a (not necessarily circular) orbit about the large one. In the polar coordinates of last page, we can write the total **energy per unit (orbiter) mass** as

$$\frac{E}{m} = \frac{\boldsymbol{v}^2}{2} - \frac{GM}{r} = \frac{{v_r}^2}{2} + \frac{{v_\eta}^2}{2} - \frac{GM}{r} \quad .$$

where the velocity has been expressed as components in the radial direction, and the direction perpendicular to this (" η "). Similarly, the **angular momentum per unit mass** is

$$\frac{L}{m} \equiv h = r v_\eta = r^2 \frac{d\eta}{dt} \quad ,$$

since in a short time dt the angle η changes by $d\eta = v_\eta dt / r$.

Elliptical orbits (continued)

In PHY 235 you will learn that this formula for the energy yields the Hamiltonian equation of motion for the orbiting particle with mass *m*. Both energy and angular momentum are conserved (thus are constant, whatever the orbit), so

$$\frac{E}{m} = \frac{{v_r}^2}{2} + \frac{h^2}{2r^2} - \frac{GM}{r} \quad .$$

But $$v_r = \frac{dr}{dt} = \frac{dr}{d\eta}\frac{d\eta}{dt} = \frac{dr}{d\eta}\frac{h}{r^2} \quad ,$$ so

$$\frac{E}{m} = \left(\frac{dr}{d\eta}\right)^2 \frac{h^2}{2r^4} + \frac{h^2}{2r^2} - \frac{GM}{r} \quad ;$$

a first-order, nonlinear, differential equation.

Elliptical orbits (continued)

You will also learn in PHY 235 that this differential equation has the unique solution

$$r = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\eta} \quad ,$$

that is, the equation of an ellipse, where the semimajor axis *a* and eccentricity ε are given in terms of the angular momentum per unit mass *h* and total energy *E* by

$$h = \sqrt{GMa\left(1-\varepsilon^2\right)} \qquad \frac{E}{m} = -\frac{GM}{2a} \quad \Rightarrow \quad a = -\frac{GMm}{2E} \qquad \varepsilon = \sqrt{1+2\left(\frac{h}{GM}\right)^2 \frac{E}{m}}$$

The algebra we won't discuss, II

But we can demonstrate that this solution works. (Actually that's the way you'll do it in PHY 235 too.) It's not hard to do, but it takes a lot of writing. First the derivative:

$$r = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\eta} \quad \Rightarrow \quad \cos\eta = \frac{a\left(1-\varepsilon^2\right)-r}{\varepsilon r}$$

$$\frac{dr}{d\eta} = -\frac{a\left(1-\varepsilon^2\right)}{\left(1+\varepsilon\cos\eta\right)^2}\frac{d}{d\eta}\left(1+\varepsilon\cos\eta\right)$$

$$= \frac{a\left(1-\varepsilon^2\right)}{\left(1+\varepsilon\cos\eta\right)^2}\varepsilon\sin\eta = \frac{r^2\varepsilon\sin\eta}{a\left(1-\varepsilon^2\right)}$$

The algebra we won't discuss, II (continued)

Then the square:

$$\left(\frac{dr}{d\eta}\right)^2 = \frac{r^4\varepsilon^2}{a^2\left(1-\varepsilon^2\right)^2}\sin^2\eta = \frac{r^4\varepsilon^2}{a^2\left(1-\varepsilon^2\right)^2}\left(1-\cos^2\eta\right)$$

$$= \frac{r^4\varepsilon^2}{a^2\left(1-\varepsilon^2\right)^2}\left(1-\frac{\left[a\left(1-\varepsilon^2\right)-r\right]^2}{\varepsilon^2 r^2}\right)$$

$$= \frac{r^4}{a^2\left(1-\varepsilon^2\right)}\left(-1+\frac{2a}{r}-\frac{a^2\left(1-\varepsilon^2\right)}{r^2}\right)$$

The algebra we won't discuss, II (continued)

$$\frac{E}{m} = \left(\frac{dr}{d\eta}\right)^2 \frac{h^2}{2r^4} + \frac{h^2}{2r^2} - \frac{GM}{r}$$

$$= \frac{h^2}{2a^2\left(1-\varepsilon^2\right)}\left(-1 + \frac{2a}{r} - \frac{a^2\left(1-\varepsilon^2\right)}{r^2}\right) + \frac{h^2}{2r^2} - \frac{GM}{r} \quad \text{, or}$$

$$0 = \left[-\frac{h^2}{2a^2\left(1-\varepsilon^2\right)} - \frac{E}{m}\right] + \left[\frac{h^2}{a\left(1-\varepsilon^2\right)} - GM\right]\frac{1}{r} + \left[\frac{h^2}{2} - \frac{h^2}{2}\right]\frac{1}{r^2}$$

So this will be true for all *r* if and only if the square-bracket terms are all zero. The last one we get for free. The other two determine *a* and ε in terms of *M* and *h:*

The algebra we won't discuss, II (continued)

$$\frac{h^2}{a\left(1-\varepsilon^2\right)} = GM \quad \text{and} \quad \frac{h^2}{2a^2\left(1-\varepsilon^2\right)} = -\frac{E}{m}$$

Divide the first by the second:

$$\frac{h^2}{a\left(1-\varepsilon^2\right)} \frac{2a^2\left(1-\varepsilon^2\right)}{h^2} = 2a = -\frac{GMm}{E} \quad \Rightarrow \quad a = -\frac{GMm}{2E}$$

Substitute back:

$$-\frac{GMm}{2E} \frac{\left(1-\varepsilon^2\right)}{h^2} = \frac{1}{GM} \quad \Rightarrow \quad \varepsilon = \sqrt{1+\frac{2Eh^2}{G^2M^2m}}$$

Both as advertised.

Elliptical orbits (continued)

Example 3. *A body of mass m is in an elliptical orbit with eccentricity ε. What is the ratio of orbital speeds at perihelion and aphelion?*

Consult the properties of ellipses (page 11):

$$r_p = a(1-\varepsilon) \qquad r_a = a(1+\varepsilon)$$

Minimum and maximum distances from the focus.

Use conservation of angular momentum:

$$mr_pv_p = mr_av_a$$

$$ma(1-\varepsilon)v_p = ma(1+\varepsilon)v_a$$

$$\boxed{\frac{v_p}{v_a} = \frac{1+\varepsilon}{1-\varepsilon}} \quad .$$

Elliptical orbits (continued)

Example 4. *What is the speed of this object, at a point for which its distance from the larger mass is r?*

Use conservation of energy:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a}$$

$$\boxed{v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}}$$

This useful result is often called the ***vis-viva* equation**.

Kepler's Laws

Before Newton discovered the laws of motion and gravity, Johannes Kepler noticed the following facts about planetary orbits, which had been measured with great accuracy by his mentor Tycho Brahe:

- Each planet follows an elliptical orbit, with the Sun at one focus.
- The line between the Sun and each planet sweeps out equal areas in equal times, wherever the planet is in its orbit.
- The square of a planet's orbital period is proportional to the cube of its orbital semimajor axis.

All of these empirical facts can be derived from Newton's laws; that is, Newton's theories are validated by Tycho's observations, as Newton himself noted. To wit:

Kepler's Laws (continued)

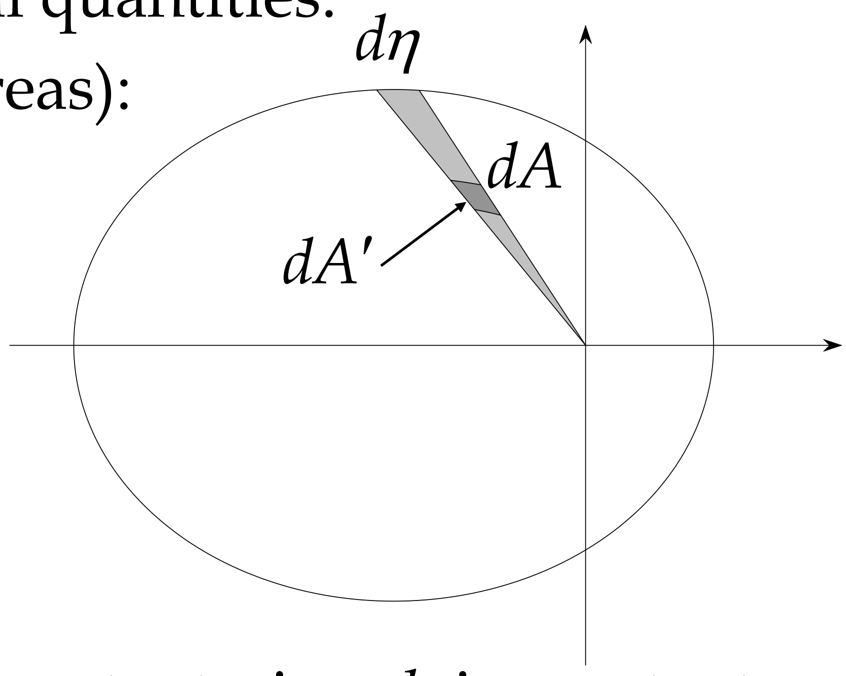
- Kepler's first law: we just showed, above, that elliptical orbits are consistent with gravity and the laws of motion, and the relationship between the parameters of the ellipse and the conserved mechanical quantities.
- Kepler's second law (equal areas):

$$dA' = dr'\left(r'd\eta\right)$$

$$dA = d\eta\int_0^r r'dr' = \frac{r^2}{2}d\eta$$

$$\frac{dA}{dt} = \frac{r^2}{2}\frac{d\eta}{dt} = \frac{1}{2}rv_\eta$$

$$= \frac{h}{2} = \frac{1}{2}\sqrt{GMa\left(1-\varepsilon^2\right)} = \text{constant, since } h \text{ is constant.}$$



Kepler's Laws (continued)

- Kepler's third law (period and semimajor axis):

$$\frac{dA}{dt} = \frac{h}{2}$$

Integrate over one orbital period:

$$\int_0^A dA' = \frac{h}{2}\int_0^P dt$$

$$A = \frac{h}{2}P = \pi ab$$

$$P^2 = \frac{4\pi^2 a^2 b^2}{h^2} = \frac{4\pi^2 a^4\left(1-\varepsilon^2\right)}{GMa\left(1-\varepsilon^2\right)} = \boxed{\frac{4\pi^2}{GM}a^3} \quad .$$

Tycho and Kepler, Newton and Copernicus

Thus the Kepler/Tycho results are consistent with the predictions of Newtonian dynamics, and can be regarded as the first experimental (pre-)validation of Newton's theories.

They also supply a crucial missing piece of the Copernican model of the solar system:

- ❑ Copernicus hypothesized circular orbits for the planets. Measurements like Tycho's were accurate enough to rule out circular heliocentric orbits.
- ❑ And the Ptolemaic geocentric theory, with ~15 epicycles per planet, was thus in much better agreement with the observations.
  - Copernicus had only one free parameter per planet: the orbital radius.

Tycho and Kepler, Newton and Copernicus (continued)

- A theory with more free parameters can always fit experimental results better than one with fewer free parameters.

❑ So, as neither theory had an explanation for why anything was orbiting anything else, a contemporary of Copernicus's could *maintain* – not just *assert* – that the Ptolemaic theory was better, as it gave better agreement with observations.

Tycho and Kepler, Pohorelec, Prague, Czech Republic.