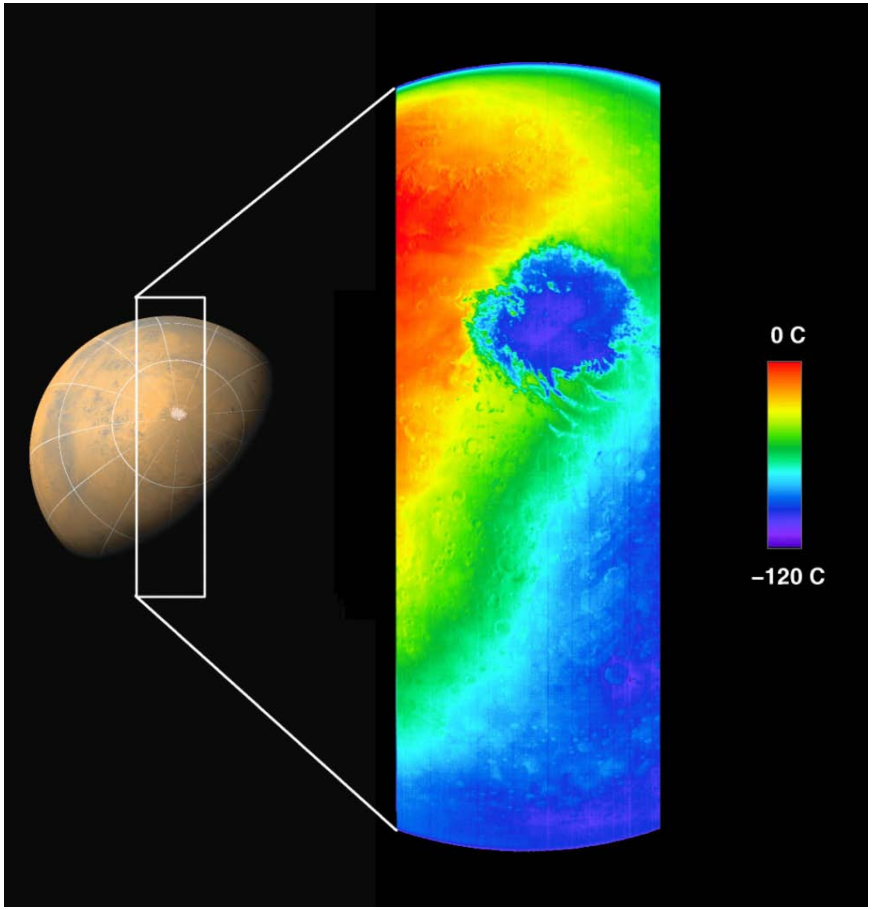
Today in Astronomy 111: more details on energy transport and the temperatures of the planets

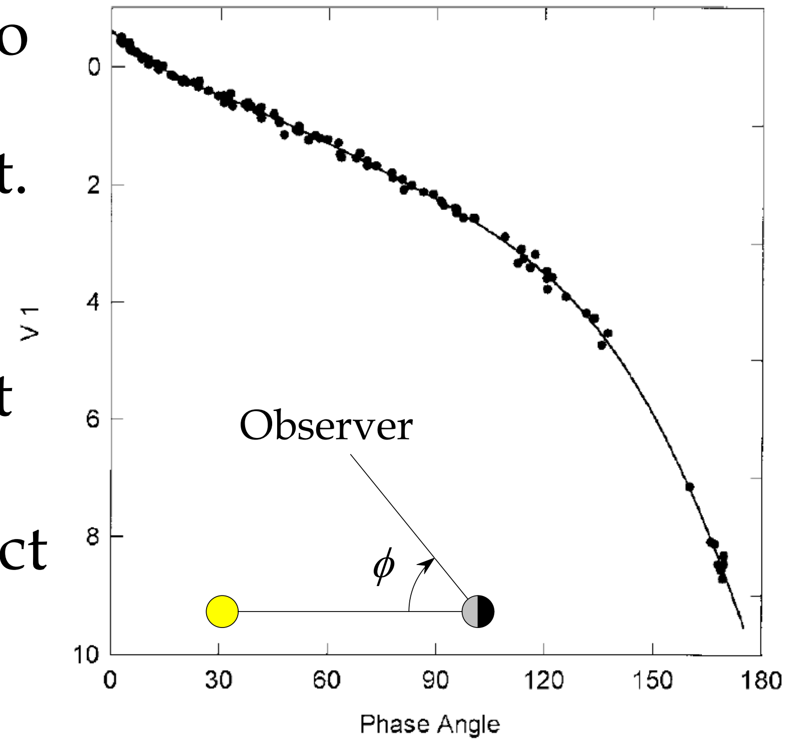
- More about albedo and emissivity
- More about the temperature of sunlit, radiation-cooled surfaces
- Heat conduction and internal heat generation
- Internal temperatures of rocky planets
- How small could differentiated bodies be?



Temperature image of Mars, made by Mars Odyssey's Thermal Emission Imaging System (ASU/JPL/NASA).

Geometric and Bond albedo

- The **geometric albedo** is the ratio of the flux reflected head-on (toward the Sun) to that incident.
- The **Bond albedo** is the ratio of the *total* flux reflected and scattered in all directions, to that incident.
  - Bumpy surfaces tend to reflect light back the way it came. The Moon and Mercury, for example, are more than 10 times brighter full than half. So their geometric and Bond albedoes are similar.



Brightness of Mercury as a function of phase angle ϕ, from SOHO (Mallama et al. 2002).

Geometric and Bond albedo (continued)

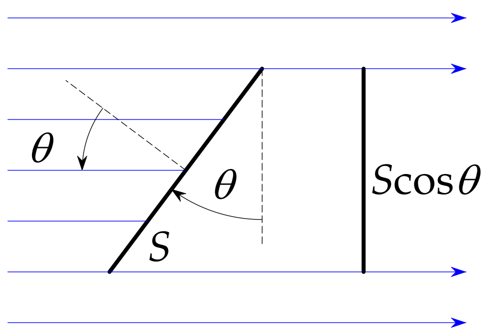
- Albedo generally varies with wavelength. At a particular wavelength the ratio of reflected and scattered light to that incident is A_λ, the **monochromatic albedo**.
- The Bond albedo, which we'll call A_b, is what one usually wants to use in solar-heating calculations.
  - Naturally, it's the one that's hardest to measure.

Planet or moon	Geometric albedo	Bond albedo
Mercury	0.106	0.119
Venus	0.65	0.75
Earth	0.367	0.306
Moon	0.12	0.11
Mars	0.15	0.25
Jupiter	0.52	0.343
Io	*0.62*	
Europa	*0.68*	
Ganymede	*0.44*	
Callisto	*0.19*	
Saturn	0.47	0.342
Titan	*0.22*	
Enceladus	*1.0*	
Uranus	0.51	0.3
Neptune	0.41	0.29
Pluto	0.6	0.5

Sunlight-heated surfaces and their temperatures

So far, we have assumed solar-heated bodies to have uniform surface temperature, but this of course isn't quite right.

Because of the high incidence angle of sunlight in the polar regions of most planets, it's colder there.



- Shadow of plane with area S has area $S\cos\theta$ with sunlight incident at angle θ. But it still emits blackbody radiation from its full area.
- Thus variation of temperature with latitude.

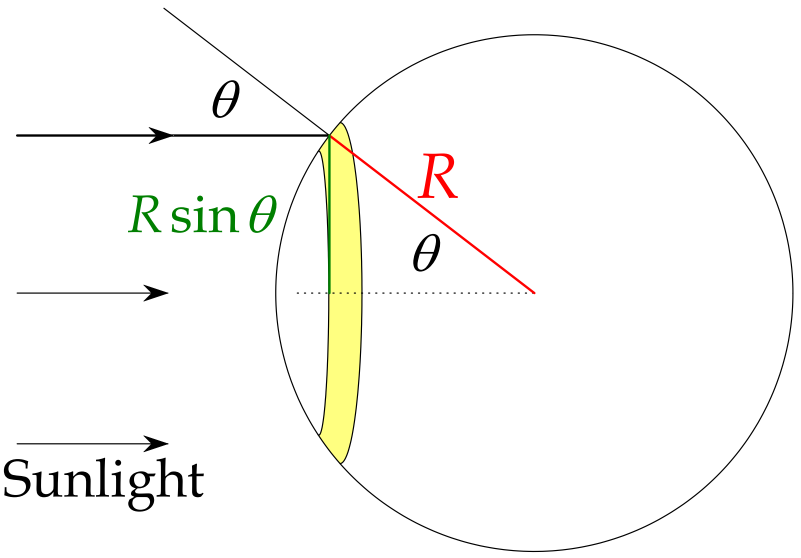
And: most slowly-rotating bodies get much warmer on the sunlit side than the dark side.

- Thus variation of temperature with angle between vertical and Sun.

Sunlight-heated surfaces and their temperatures (continued)

Example: a slow rotator. Suppose a planet with no atmosphere, in circular orbit 1 AU from a star just like the Sun, and with uniform albedo, rotates with period equal to its orbital period, so that it always shows the same face to the Sun. Neglecting the conduction of heat, what is the distribution of temperature on its sunlit side?

Consider a ribbon of the surface at an angle θ from the sub-solar point, as seen from the center of the planet, and with infinitesimal angular width $d\theta$.



Sunlight-heated surfaces and their temperatures (continued)

Every point on the ribbon, then, receives the same flux of sunlight. The area of the ribbon, on the surface, is

$$dS = \left(2\pi R \sin\theta\right)\left(R d\theta\right) = 2\pi R^2 \sin\theta d\theta \quad ,$$

and the projected area, perpendicular to the direction of sunlight – that is, the area of the ribbon's shadow – is

$$dS_\perp = dS \cos\theta = 2\pi R^2 \cos\theta \sin\theta d\theta \quad .$$

Thus the power absorbed by the ribbon is

$$dP_{in} = \left(1 - A_b\right) f_\odot dS_\perp = \left(1 - A_b\right) \frac{L_\odot}{4\pi r^2} 2\pi R^2 \cos\theta \sin\theta d\theta \quad .$$

Sunlight-heated surfaces and their temperatures (continued)

Meanwhile, the ribbon is emitting blackbody radiation, in the amount

$$dP_{out} = f_{\mathrm{ribbon}} dS = \varepsilon\sigma T_s^4 2\pi R^2 \sin\theta d\theta \quad .$$

So if the ribbon is in equilibrium, its temperature is given by

$$dP_{in} = dP_{out}$$

$$\left(1 - A_b\right)\frac{L_\odot}{4\pi r^2} 2\pi R^2 \cos\theta \sin\theta d\theta = \varepsilon\sigma T_s^4 2\pi R^2 \sin\theta d\theta$$

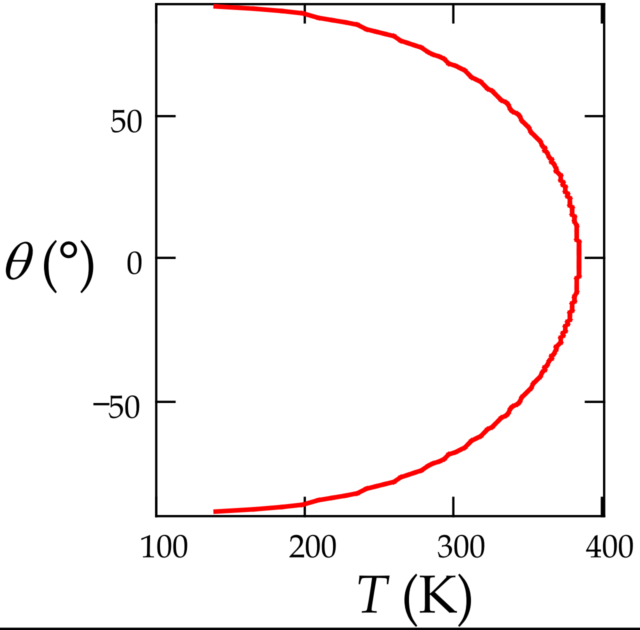
$$T_s = \left(\frac{1 - A_b}{\varepsilon}\frac{L_\odot}{4\pi\sigma r^2}\cos\theta\right)^{1/4} = T_0\left(\cos\theta\right)^{1/4}$$

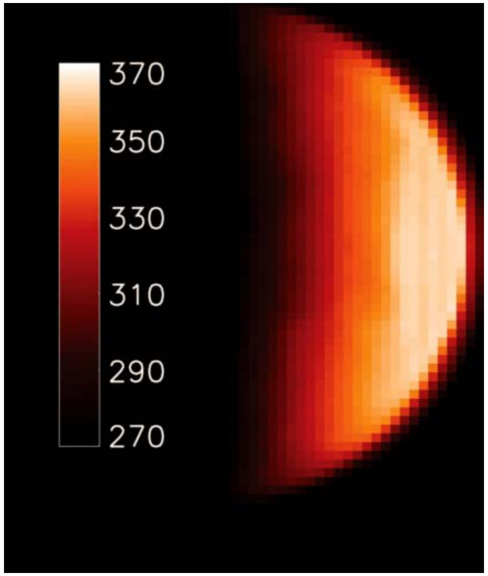
Sunlight-heated surfaces and their temperatures (continued)

Using parameters of the Moon (Bond albedo 0.11, emissivity 1),

$$T_0 = \left(\frac{1 - A_b}{\varepsilon} \frac{L_\odot}{4\pi\sigma r^2} \right)^{1/4} = 382\ \mathrm{K} \quad .$$

Not far from the temperature at the Moon's subsolar point!





Data from Jessica Sunshine and the EPOXI team.

Energy transport in planets

Energy is transported primarily by

- conduction,
- radiation, or
- convection.

Usually one mechanism dominates.

- Transport in solids is usually dominated by conduction.
- Radiation usually dominates in space and tenuous gases.
- Convection and radiation are usually most important in the interiors of stars, and in planetary atmospheres and cores, but conduction is often significant.
- Conduction is most often applicable in terrestrial-planetary interiors, which we will now discuss.

Heat conduction

Heat conduction is the transport of energy by collisions between particles (in a gas or an electrical conductor), or by exchange of lattice vibrations (in an insulating solid).

- The rate at which heat flows is called the **heat flux,** f_T , which like radiation flux has the units erg sec^{-1} cm^{-2}.
- Definition: if two planar, uniform-temperature surfaces are separated by infinitesimal distance *dz* and are infinitesimally different in temperature by *dT,* and the medium separating the planes is uniform, then the heat flux through the surfaces is

$$f_T(z) = -\kappa_T(z)\frac{dT}{dz} \quad .$$

where κ_T is called the **thermal conductivity.**

Heat conduction (continued)

- In three dimensions we would need to make the heat flux a vector, and speak of the gradient instead of a derivative:

$$\boldsymbol{f}_T(\boldsymbol{r}) = -\kappa_T(\boldsymbol{r})\nabla T \quad .$$

- But the radial component of the gradient in spherical coordinates is just *d/dr*, so for spherical symmetry the flux is radial and has magnitude

$$f_T(r) = -\kappa_T(r)\frac{dT}{dr} \quad .$$

Thermal conductivity of common materials at T = 300 K:

Material	κ_T (10^5 erg sec^{-1} cm^{-1} K^{-1})
Quartz	3
Basaltic rock	4.5
Stainless steel	16

Internal heat generation

We have already discussed two sources of internal heat in planetary bodies, accretion heating and radioactive heating, in lecture on 20 and 22 September.

- Radioactive heating is relevant for planetary interiors today.
  From table shown on 22 September: the radioactive heating rate per unit mass, $\Lambda = \Lambda_{rad}$, of carbonaceous chondrite meteoritic material is

$$\begin{aligned} \Lambda &= 5.52 \times 10^{-8} \text{ erg sec}^{-1} \text{ gm}^{-1} \quad \text{today;} \\ &= 5.26 \times 10^{-2} \text{ erg sec}^{-1} \text{ gm}^{-1} \quad \text{4.6 Gyr ago.} \end{aligned}$$

Internal heat generation (continued)

Consider the power dQ generated inside a spherically symmetric object; specifically, the heat generated within a spherical shell with radius r and thickness dr: if the mass density is ρ,

$$dQ = \Lambda dm = \Lambda\rho \times 4\pi r^2 dr$$

or

$$\frac{dQ}{dr} = 4\pi r^2 \rho\Lambda.$$

But **if the temperature is constant**, as much heat must flow out of this shell as is generated there:

$$\frac{dQ}{dr} = \frac{d}{dr}\left(4\pi r^2 f_T\right) = -\frac{d}{dr}\left(4\pi r^2 \kappa_T \frac{dT}{dr}\right) = 4\pi r^2 \rho\Lambda$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = -\frac{\rho\Lambda}{\kappa_T} \quad .$$

Poisson's equation (for spherical symmetry)

The AST 111 calculus palette, page 2

To solve differential equations, need as many **boundary conditions** – values of the solution – as the order of differential equation, to evaluate integration constants.

First order: $\frac{d}{dx} f(x) = g(x) \quad \Leftrightarrow \quad f(x) = \int g(x)\,dx = G(x) + C$

$$C = f(x_0) - G(x_0)$$

Second order: for example, $\frac{d^2}{dx^2} f(x) = g(x) \quad \Leftrightarrow$

$$f(x) = \int \left(\int g(x)\,dx \right) dx = \int \big(G(x) + C \big)\,dx = \Gamma(x) + Cx + D$$

$$\left.\begin{aligned} f(x_0) &= \Gamma(x_0) + Cx_0 + D \\ f(x_1) &= \Gamma(x_1) + Cx_1 + D \end{aligned}\right\} \quad \text{two equations in the unknowns } C \text{ and } D.$$

Temperature of the interior of a rocky planet

Poisson's equation is a differential equation we can solve for T, given a planet with mass M and a prescription for the density ρ, heating rate Λ, and thermal conductivity κ_T.

It is a second-order differential equation, so we need two boundary conditions, conveniently provided by

- the surface temperature, set by solar heating and the *total* radioactive heating power $P_{\mathrm{rad}} = M\Lambda$ (see lecture, 22 September), here assumed uniform and modified for non-blackbodies:

$$T_s = \left(\frac{(1-A)}{\varepsilon} \frac{L_\odot}{16\pi\sigma r^2} + \frac{M\Lambda}{4\pi\sigma\varepsilon R^2} \right)^{1/4}$$

- …and the fact that the central temperature must be finite.

Temperature of the interior of a rocky planet (continued)

Let us assume for simplicity a rocky solar-system body (mass M, radius R) with **uniform** density ρ, and take the thermal conductivity κ_T to be independent of temperature. Integrate the Poisson equation twice, and apply the boundary conditions:

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = -\frac{\rho\Lambda}{\kappa_T}r^2$$

$$\int\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)dr = -\frac{\rho\Lambda}{\kappa_T}\int r^2 dr$$

$$r^2\frac{dT}{dr} = -\frac{\rho\Lambda}{\kappa_T}\frac{r^3}{3} + C$$

Temperature of the interior of a rocky planet (continued)

$$\int \frac{dT}{dr} dr = -\frac{\rho \Lambda}{3\kappa_T} \int r dr + C \int \frac{dr}{r^2}$$

$$T = -\frac{\rho \Lambda}{6\kappa_T} r^2 - \frac{C}{r} + D \quad ,$$

where C and D are integration constants.

- At $r = 0$: T approaches infinity unless $C = 0$.
- At $r = R$: $T = T_s = -\frac{\rho \Lambda}{6\kappa_T} R^2 + D \quad \Rightarrow \quad D = T_s + \frac{\rho \Lambda}{6\kappa_T} R^2$.

Thus

$$\boxed{T(r) = T_s + \frac{\rho \Lambda}{6\kappa_T}\left(R^2 - r^2\right) \quad .}$$

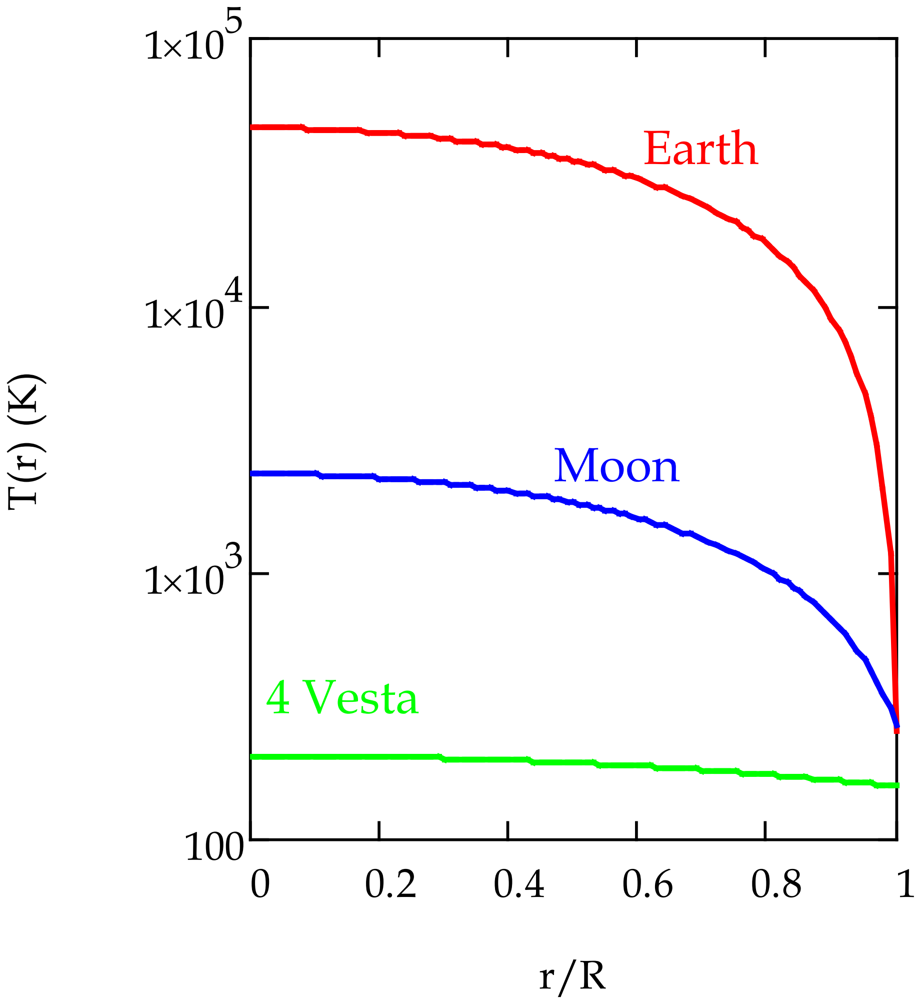
Temperature of the interior of a rocky planet (continued)

Taking density to be $\rho = 3M/4\pi R^3$; the heating rate to be that in carbonaceous chondrites today, $\Lambda = 5.52\times10^{-8}$ erg sec^{-1} gm^{-1}; and thermal conductivity $\kappa_T = 4.47\times10^{5}$ erg sec^{-1} cm^{-1}K^{-1}, as appropriate for silicate rocks; we get:

Body	Orbital radius *r*, AU	Mass *M*, gm	Radius *R*, km	Albedo A_b	$T(R)$, K	$T(0)$, K
Earth	1.00	5.98×10^{27}	6380	0.31	254	46300
Moon	1.00	7.35×10^{25}	1740	0.11	270	2350
4 Vesta	2.36	2.59×10^{23}	265	0.42	158	206

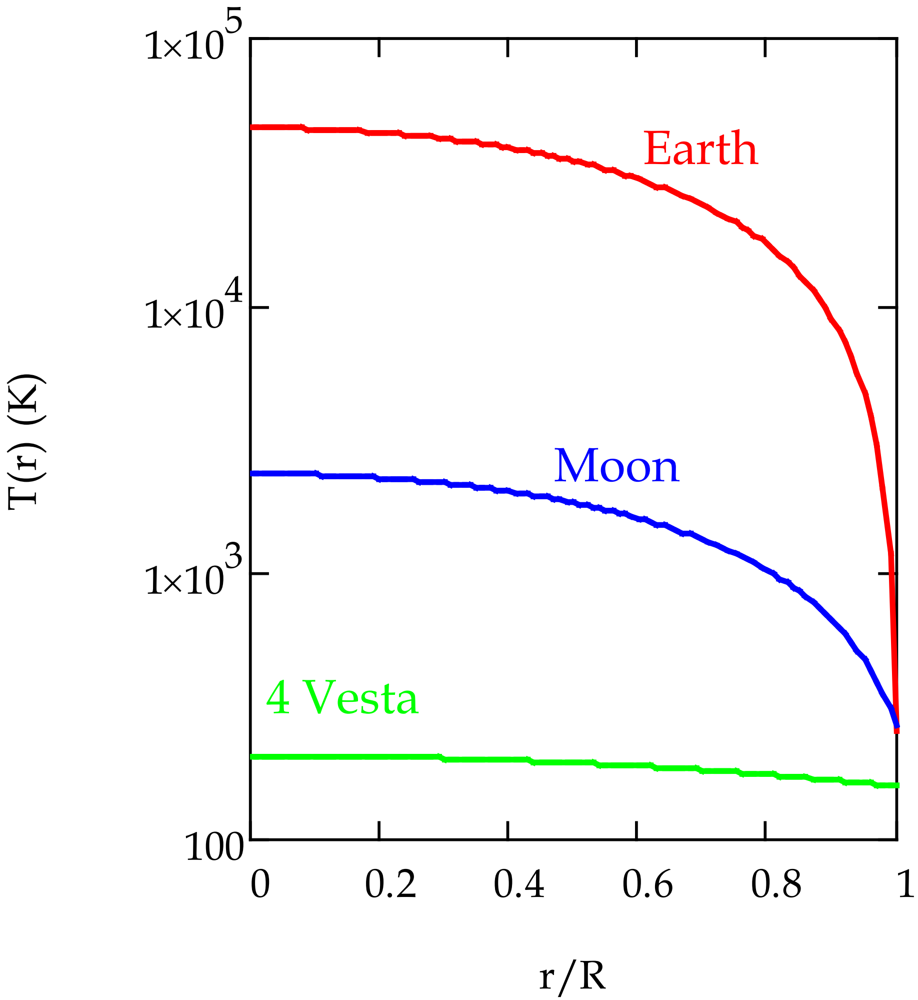
$T(0)$ too high for Earth and Moon; about right for Vesta.

Temperature of the interior of a rocky planet (continued)



- Remember, each body is considered uniform in density here.
- When temperatures exceed 2000 K, they are overestimates: κ_T increases linearly with increasing T for liquid metals…
- …and **convection** is important for heat transport in liquids, too.

Temperature of the interior of a rocky planet (continued)



Nevertheless this demonstrates a few important points:

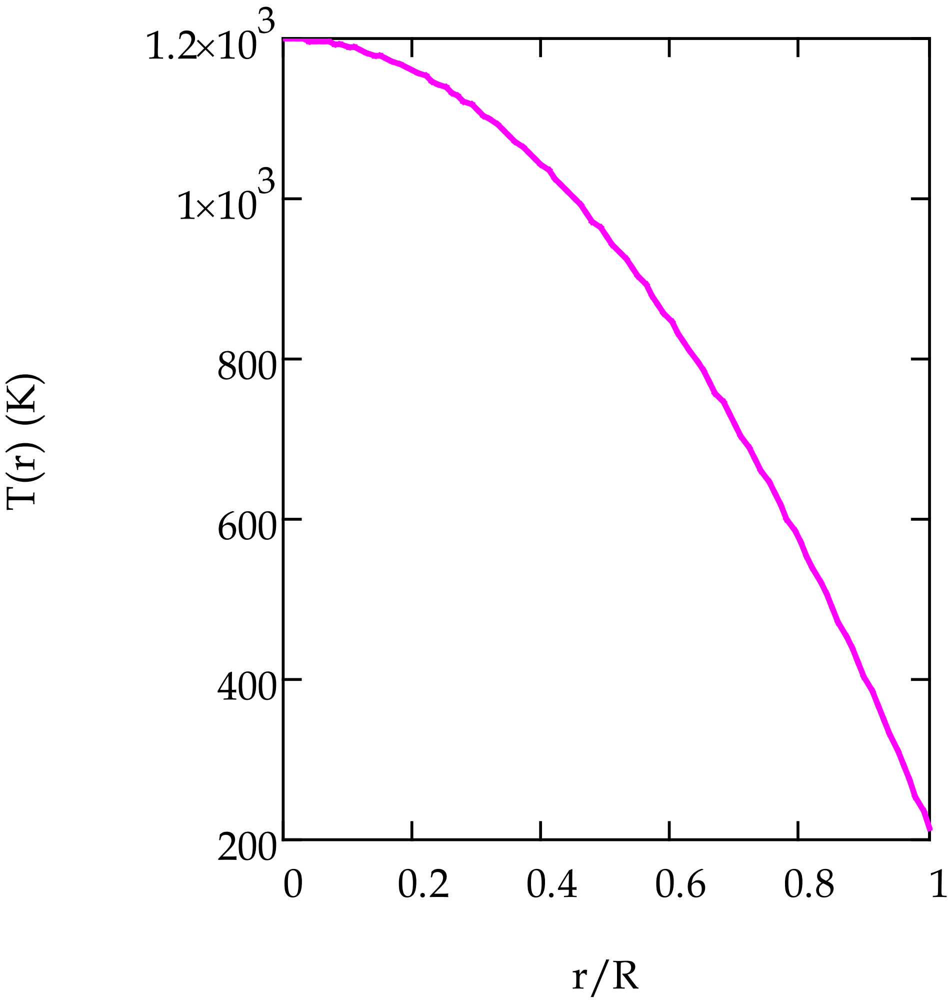
- If Earth weren't already differentiated, it would become so very quickly.
- If the Moon has any liquid metal in its core, it's just barely liquid.
- Vesta is solid through and through, and probably has been for quite some time.

The smallest differentiated bodies

What about earlier times?

- At the time of CAI formation 4.568 Gyr ago, the radioactive heating power and proto-Solar luminosity were
$$\Lambda = 5.26 \times 10^{-2} \text{ erg sec}^{-1} \text{ gm}^{-1},$$
$$L = 2.5 L_{\odot} = 9.57 \times 10^{33} \text{ erg sec}^{-1}.$$
- Consider a small uniform sphere with non-porous, carbonaceous-chondrite composition, in an orbit like that of 1 Ceres: $\rho = 2.7 \text{ gm cm}^{-3}, A_b = 0.05, r = 2.77 \text{ AU}$...
- …and suppose it is just barely massive enough that mafic minerals melt in its center: $T(0) = 1200$ K, so that on the average $\kappa_T = 4.30 \times 10^{5} \text{ erg sec}^{-1} \text{ cm}^{-1} \text{K}^{-1}$.

The smallest differentiated bodies (continued)



Solving (iteratively) for the R which gives $T(0)$ = 1200 K, we get

$$R = 1.34\times10^{5}\ \text{cm}$$

$$M = 2.70\times10^{16}\ \text{gm}$$

$$T_s = 216\ \text{K}$$

Thus it is possible that nonporous bodies as small as a few km in size melted in their centers and became differentiated, if they formed early enough in the Solar system's history.