Astronomy 111 Recitation #4

22-23 September 2011

Formulas to remember

Astronomy 111 Math Palette 1

Fundamental theorem of calculus: if \( f(x) = \frac{d}{dx} F(x) \), then
\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a).
\]

\[
\frac{d}{dx} x^n = nx^{n-1} \quad \Leftrightarrow \quad \int_{a}^{b} x^n \, dx = \frac{1}{n+1} \left( b^{n+1} - a^{n+1} \right)
\]

\[
\frac{d}{dx} e^{cx} = ce^{cx} \quad \Leftrightarrow \quad \int_{a}^{b} e^{cx} \, dx = \frac{1}{c} (e^{cb} - e^{ca})
\]

\[
\frac{d}{dx} \sin cx = c \cos cx \quad \Leftrightarrow \quad \int_{a}^{b} \cos cx \, dx = \frac{1}{c} (\sin cb - \sin ca)
\]

\[
\frac{d}{dx} \cos cx = -c \sin cx \quad \Leftrightarrow \quad \int_{a}^{b} \sin cx \, dx = -\frac{1}{c} (\cos cb - \cos ca)
\]

Angular momentum \( (L) \) and moment of inertia \( (I) \)

In general for a point mass \( m \) with velocity \( v \):
\[
L = r \times p = r \times m v = (m v r \sin \alpha) \hat{n} = m v r \perp \hat{n} = I \omega \hat{n}.
\]

For a point mass \( m \) revolving in a circle with radius \( r_{\perp} \) at speed \( v \):
\[
L = m v r_{\perp} = m r_{\perp}^2 \frac{v}{r_{\perp}} = I \omega, I = m r_{\perp}^2.
\]

For a infinitesimal element of volume in spherical coordinates, with density \( \rho \), that is part of a sphere rotating about the \( z \) axis:
\[
dl = r_{\perp}^2 dm = (r \sin \theta)^2 \rho r^2 \sin \theta dr d\theta d\phi = \rho r^4 \sin \theta \sin \theta d\theta d\phi.
\]

For a sphere with radius \( R \) and density \( \rho(r) \) that is a function only of \( r \) (not \( \theta \) or \( \phi \)):
\[
I = \iiint_{\text{whole sphere}} dm r_{\perp}^2 = \frac{2\pi}{3} \int_{0}^{R} \rho(r) r^4 \, dr = \frac{8\pi}{3} \int_{0}^{R} \rho(r) r^4 \, dr
\]

\[
M = \iiint_{\text{whole sphere}} dm = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{R} \rho(r) r^2 \, dr \, d\theta \, d\phi = 4\pi \int_{0}^{R} \rho(r) r^2 \, dr
\]

Surface temperature of a blackbody planet with internal radioactive heating
\[
\frac{L}{4\pi r^2} = \pi R^2 + M \Lambda_{\text{rad}}(t) = 4\pi R^2 \sigma T_s^4 \quad \Leftrightarrow \quad T_s = \left( \frac{L + 4M \Lambda_{\text{rad}}(t) r^2 / R^2}{16\pi \sigma r^2} \right)^{1/4},
\]

where the planet is assumed to be perfectly absorbing, and where \( \Lambda_{\text{rad}}(t) \) is the power per unit mass as a function of time, generated by all the significant radionuclides (see lecture notes for 20 September).

**Workshop problems (do after discussing Homework #3):**

**Warning!** The workshop problems you will do in groups in Recitation are a crucial part of the process of building up your command of the concepts important in AST 111 and subsequent courses. Do not, therefore, do your work on scratch paper and discard it. Better for each of you to keep your own account of each problem, in some sort of bound notebook.

1. **A simple differentiated spherical body.** A planet with mass \( M \) and radius \( R \) has a uniform-density core with density \( \rho \) and radius \( 2R/3 \), and a uniform-density mantle and crust with density \( 2\rho/5 \) for \( r = 2R/3 \) to \( R \). What is the core density \( \rho \), in terms of \( M \) and \( R \)?

2. What is the moment of inertia of the sphere in problem 1, in terms of \( M \) and \( R \)?

3. Compare this simple model to the measured moment of inertia of the three innermost terrestrial planets, and to the densities of common mineral and metal ingredients of planets (c.f. lecture on 13 September, and the How Big Is That sheet). What do you think their cores and mantles are likely to be made of, and how large are their cores?

4. **Radioactive heating and planet size.** A planet lies \( r = 1 \) AU from a star with luminosity \( L = 1L_\odot \). Its average density is the same as Earth’s, 5.515 gm cm\(^{-3}\). Its composition is such that the radioactive heating rates through its history are as described in the table given in lecture on 23 September, page 3. Its surface temperature is twice as large as it would be if heated only by starlight. It can be treated as a blackbody for the purposes of heating by starlight.
   a. Suppose it is newborn, and that accretional heating is unimportant. What is its radius, in Earth radii?
   b. Suppose it is 4.567 billion years old. What is its radius, in Earth radii? (Is this possible?)

**Learn your way around the sky, lesson 4.** (An exclusive feature of AST 111 recitations.) Use the lab’s celestial globes, and a Web browser, to help answer these questions about the celestial sphere and the constellations.

5. What is the sidereal time right now – that is, what is the RA of stars on the meridian? First, estimate it crudely by noting how long it has been since an equinox (one of which happened last night at 11:09 PM EDT!). Then look it up: google “Sidereal time calculator” to get yourself to the online calculator provided by Our Nation’s Timekeeper, the US Naval Observatory.

6. In Recitation #2, you considered a star with right ascension and declination \( \alpha \) and \( \delta \); drew a Cartesian coordinate system with the \( x \)-axis pointing through \( \alpha = \delta = 0 \) and the \( z \)-axis pointing toward \( \delta = 90^\circ \); and derived an an expression for a unit vector pointing at the star, in terms of the Cartesian unit vectors \( \hat{x}, \hat{y}, \) and \( \hat{z} \). You got
\[ \hat{n} = \cos \delta \cos \alpha \hat{x} + \cos \delta \sin \alpha \hat{y} + \sin \delta \hat{z} \]

a. In the same coordinate system, construct a unit vector that points toward the zenith, and depends upon sidereal time $ST$ and latitude $\lambda$.

b. Derive an expression for the angle between this new unit vector, and the old one above. Look for a simplification that allows you to express the result in terms of the difference between $ST$ and the right ascension of the object. This new angle is called the zenith angle (ZA); the right-angle complement, the altitude, is the angular distance between the star and the horizon. The difference between $ST$ and RA is called the hour angle (HA).

7. Match the celestial object to the constellation to which it belongs.

<table>
<thead>
<tr>
<th>Objects</th>
<th>Constellations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brightest star in the sky</td>
<td>Andromeda</td>
</tr>
<tr>
<td>Most massive, luminous star in the galaxy</td>
<td>Canis Major</td>
</tr>
<tr>
<td>Nearest region of ongoing star formation</td>
<td>Carina</td>
</tr>
<tr>
<td>Nearest region of ongoing, massive (O-B) star formation</td>
<td>Centaurus</td>
</tr>
<tr>
<td>Nearest open stellar cluster</td>
<td>Hercules</td>
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<td>Nearest globular stellar cluster</td>
<td>Ophiuchus</td>
</tr>
<tr>
<td>Center of the Milky Way Galaxy</td>
<td>Orion</td>
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<td>Nearest spiral galaxy to Milky Way</td>
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<td>Nearest active galaxy (supermassive black hole, relativistic jets)</td>
<td>Taurus</td>
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<td>Nearest rich cluster of galaxies</td>
<td>Virgo</td>
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</table>
Problem solutions

1. 

\[ M = 4\pi \left[ \rho \int_0^{2R/3} r^2 dr + \frac{2\rho}{5} \int_{2R/3}^{R} r^2 dr \right] = 4\pi \rho \left[ \frac{r^3}{3} \bigg|_0^{2R/3} + \frac{2}{5} \frac{r^3}{R} \bigg|_{2R/3}^{R} \right] \]

\[ = \frac{4\pi \rho R^3}{3} \left( \frac{2}{3} + \frac{2}{5} \left( 1 - \frac{2}{3} \right) \right) = \frac{4\pi \rho R^3}{3} \left( \frac{8}{27} + \frac{2}{5} \frac{19}{27} \right) = \frac{104\pi \rho R^3}{135} \text{, or} \]

\[ \rho = \frac{135M}{104\pi R^3} \]

2. 

\[ I = \frac{8\pi}{3} \left[ \rho \int_0^{2R/3} r^4 dr + \frac{2\rho}{5} \int_{2R/3}^{R} r^4 dr \right] = \frac{8\pi \rho R^5}{3} \left( \frac{r^5}{5} \bigg|_0^{2R/3} + \frac{2}{5} \frac{r^5}{R} \bigg|_{2R/3}^{R} \right) \]

\[ = \frac{8\pi \rho R^5}{15} \left( \frac{2}{3} + \frac{2}{5} \left( 1 - \frac{2}{3} \right) \right) = \frac{8\pi \rho R^5}{15} \left[ \frac{32}{243} + \frac{2}{5} \frac{211}{243} \right] = \frac{1552\pi \rho R^5}{6075} \]

\[ = \frac{1552\pi R^5}{6075} \frac{135M}{104\pi R^3} = \frac{194}{585} MR^2 = 0.33 MR^2 \]

3. The most abundant silicates are the mafic minerals enstatite and forsterite, both of which have density near 3.2 gm cm\(^{-3}\). The most abundant metal is iron, which has density 7.9 gm cm\(^{-3}\). Thus silicates and iron have a density ratio of about 2/5, as was assumed for the materials in problems 1 and 2. The inner three terrestrial planets all have \( I/ MR^2 \approx 0.33 \), as does the hypothetical planet in problems 1 and 2. Thus it is reasonable for these planets to have iron cores, with radii about two-thirds of their planets, and silicate mantles.

4. If radioactive heating doubles the surface temperature compared to starlight heating alone, then this determines the radius of the planet:

\[ T = \left( \frac{L_\odot + \frac{4M\Lambda_{rad}(t)r^2}{R^2}}{16\pi\sigma r^2} \right)^{1/4} = 2 \left( \frac{L_\odot}{16\pi\sigma r^2} \right)^{1/4} \]

\[ \frac{L_\odot + 4\frac{M\Lambda_{rad}(t)r^2}{R^2}}{16\pi\sigma r^2} = 16 \frac{L_\odot}{16\pi\sigma r^2} \]

\[ 4\frac{\pi}{3} R^3 \rho\Lambda_{rad}(t) \frac{r^2}{R^2} = 15L_\odot \]

\[ R = \frac{45L_\odot}{16\pi r^2 \rho\Lambda_{rad}(t)} \]
a. When new, the heating is dominated by the decay of $^{60}$Fe, $^{36}$Cl and $^{26}$Al, and $\Lambda_{\text{rad}} = 0.053 \text{ erg sec}^{-1} \text{ gm}^{-1}$. Thus $R = 5.3 \times 10^7 \text{ cm} = 0.083 R_{\oplus}$.

b. At age 4.567 Gyr, $\Lambda_{\text{rad}} = 5.52 \times 10^{-8} \text{ erg sec}^{-1} \text{ gm}^{-1}$, and $R$ works out to $R = 5.0 \times 10^{13} \text{ cm} = 7.9 \times 10^4 R_{\oplus} = 3.4 \text{ AU}$. We have thus run into an inconsistency, requiring that the planet’s radius be larger than the distance to the star; at current heating rates it’s not possible for radioactivity to double the surface temperature of a terrestrial planet.

5. Everybody will do this at a different time; let’s suppose it is now 23 September at 4PM EDT. Very crudely: since the autumnal equinox happened a few hours ago, the sidereal time at midnight (standard time, i.e. 1AM EDT) will be close to zero. So the current sidereal time is close to the number of hours since midnight on 23 September, modulo 24 hours. 4PM EDT on 23 September is nine hours earlier, so $ST \approx 15^h$, within a several minutes. (On the day of the autumnal equinox, wall-clock time is within several minutes of local sidereal time, all day!) A little less crudely: since the equinox occurred so close to our local midnight, the sidereal time at midnight (standard time; 1AM EDT) must have been within a few minutes of $ST = 0^h$. Now it’s 15 hours later in wall-clock time. As you showed in recitation last week, the sidereal clock runs faster than the one on the wall by a factor of $1/0.99727$, so the sidereal time $ST^h$ must be close to $ST = 15^h/0.99727 = 15.04^h = 15^h 02.5^m$.

6. a. The right ascension and declination of the zenith are $\alpha = ST, \delta = \lambda$. Thus it follows trivially from the previous result:

$$\hat{n}_0 = \cos \lambda \cos ST \hat{x} + \cos \lambda \sin ST \hat{y} + \sin \lambda \hat{z}$$

b. The dot product of the unit vectors toward zenith and object gives the zenith angle:

$$\cos ZA = \hat{n} \cdot \hat{n}_0 = \cos \delta \cos \alpha \cos \lambda \cos ST + \cos \delta \sin \alpha \cos \lambda \sin ST + \sin \delta \sin \lambda$$

$$= \left( \cos \alpha \cos ST + \sin \alpha \sin ST \right) \cos \delta \cos \lambda + \sin \delta \sin \lambda$$

$$= \cos \left( \alpha - ST \right) \cos \delta \cos \lambda + \sin \delta \sin \lambda = \cos HA \cos \delta \cos \lambda + \sin \delta \sin \lambda$$

$$ZA = \arccos \left( \cos HA \cos \delta \cos \lambda + \sin \delta \sin \lambda \right)$$

Note that if the object is on the meridian ($HA = 0$),

$$ZA = \arccos \left( \cos \delta \cos \lambda + \sin \delta \sin \lambda \right) = \arccos \left( \cos \left[ \delta - \lambda \right] \right) = \left| \delta - \lambda \right|,$$

as is perhaps intuitively obvious.

7.

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</tr>
<tr>
<td>Most massive, luminous star in the galaxy ($\eta$ Carinae)</td>
<td>Carina</td>
</tr>
<tr>
<td>Nearest region of ongoing star formation ($\rho$ Ophiuchi Cloud)</td>
<td>Ophiuchus</td>
</tr>
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<td>Objects</td>
<td>Constellations</td>
</tr>
<tr>
<td>------------------------------------------------------------------------</td>
<td>----------------</td>
</tr>
<tr>
<td>Nearest region of ongoing, massive (O-B) star formation (Orion Molecular Cloud, associated with the Orion Nebula, M42)</td>
<td>Orion</td>
</tr>
<tr>
<td>Nearest open stellar cluster (the Hyades)</td>
<td>Taurus</td>
</tr>
<tr>
<td>Nearest globular stellar cluster (the Great Hercules Cluster, M13)</td>
<td>Hercules</td>
</tr>
<tr>
<td>Center of the Milky Way Galaxy (Sagittarius A*)</td>
<td>Sagittarius</td>
</tr>
<tr>
<td>Nearest spiral galaxy to Milky Way (Andromeda Nebula, M31)</td>
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</tr>
<tr>
<td>(Centaurus A, NGC 5128)</td>
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