# Astronomy 111 Recitation #7

13-14 October 2011

**Formulas to remember**

***Albedo, emissivity, and solar/radioactive heating of Solar-system bodies***

With $A$ as the albedo at UV and visible wavelengths, $\varepsilon$ as the emissivity at mid-infrared wavelengths, and $\Lambda_{\text{rad}}$ the radioactive heating power per unit mass, the surface temperature $T_s$ of a uniform-temperature sphere with mass $M$ and radius $R$ a distance $r$ from the Sun (luminosity $L_\odot$) is

$$T_s = \left(\frac{L_\odot}{16\pi\sigma r^2}\right)^{1/4}$$

if the sphere is a blackbody: $A = 0, \varepsilon = 1$; and if radioactive heating is insignificant (lecture notes, 8 September).

$$T_s = \left(\frac{L_\odot + 4M\Lambda_{\text{rad}} r^2 / R^2}{16\pi\sigma r^2}\right)$$

if the sphere is a blackbody, but accounting for radioactive heating (lecture notes, 20 September).

$$T_s = \left(\frac{1-A}{\varepsilon}\frac{L_\odot}{16\pi\sigma r^2}\right)^{1/4}$$

as you will show below.

You should be prepared to derive these formula, and similar ones.

***Cross section*** is the effective area of the *shadow* of an object, which determines how much of an incident stream of light (or other particles) are reflected, absorbed or scattered. In this course, cross section is the geometrical area of the shadow, which for a spherical object of radius $R$ is $\pi R^2$. Note, however, that the cross section would tend to be different from the geometrical value if the object were similar in size or smaller than the wavelength of incident light, or if wavelengths are used at which the object is strongly reflective or absorbing. This idea of the cross section is used in many fields of physics and you will encounter it in several classes.

***Vocabulary note.*** If a quantity is "constant," its value doesn't vary as time varies. If it's "uniform" its value doesn't vary as position varies.

**Workshop problems (do after discussing Practice Exam #1):**

**Warning!** The workshop problems you will do in groups in Recitation are a crucial part of the process of building up your command of the concepts important in AST 111 and subsequent courses. Do not, therefore, do your work on scratch paper and discard it. Better for each of you to keep your own account of each problem, in some sort of bound notebook.

1. Most of the sunlight or starlight absorbed by a planet is at visible and ultraviolet wavelengths. Most of the blackbody radiation that it emits is emitted at mid-infrared wavelengths. If the planet has albedo $A$ and emissivity $\varepsilon$, the amount of light it absorbs is reduced by the factor $1-A$ and the amount of light that it emits is reduced by the factor $\varepsilon$

   A spherical planet has radius $R$, albedo $A$ and infrared emissivity $\varepsilon$, and lies in a circular orbit of radius $r$ about a star with luminosity $L$. Heating from internal radioactivity is negligible. Derive a formula for the surface temperature of the surface of this planet, considered uniform.

2. Two spherical planets lie the same distance away (1 AU) from a star $\left(1L_\odot\right)$. Each has an albedo that is the same *at all wavelengths;* that is, $\varepsilon = 1 - A$ no matter what the wavelength. But their appearances are very different: one has albedo 0.96, the other 0.04. By how much (in K) do their temperatures differ?

3. The two most common isotopes of uranium, $^{235}\mathrm{U}$ and $^{238}\mathrm{U}$, are both radioactive, and decay by chains of transitions to stable isotopes of lead:

$$^{235}\mathrm{U} \rightarrow {}^{207}\mathrm{Pb} + 7\,{}^{4}\mathrm{He} + 4e^{-} + 4\bar{\nu}_e + 7.24\times10^{-5}\ \mathrm{erg}$$
$$^{238}\mathrm{U} \rightarrow {}^{206}\mathrm{Pb} + 8\,{}^{4}\mathrm{He} + 6e^{-} + 6\bar{\nu}_e + 7.59\times10^{-5}\ \mathrm{erg}$$

Only the first step of each decay sequence has a long half-life:

$$\lambda_{235} = 9.849\times10^{-10}\ \mathrm{year}^{-1} \quad t_{1/2}(235) = 0.7038\ \mathrm{Gyr}$$
$$\lambda_{238} = 1.551\times10^{-10}\ \mathrm{year}^{-1} \quad t_{1/2}(238) = 4.468\ \mathrm{Gyr}$$

and the rest of the steps are so much shorter that they can be considered instantaneous on the time scales of geology and the Solar system. A non-radiogenic isotope of lead, $^{204}\mathrm{Pb}$, provides a convenient stable daughter-species reference.

One problem with using either of these decay chains by itself for age measurements, in the manner of Rb-Sr, is that the dynamics of U and Pb in minerals are different enough that diffusion and chemical effects can alter the relative abundances of the parent and daughter nuclides significantly, apart from any effect of radioactivity on these abundances. Geologists and meteoriticists use the two chains in tandem to account for these effects. Here's how.

a. Show (by deriving a formula) that a plot of the present value of the two daughter abundances, $D_{235} = {}^{207}\mathrm{Pb}/{}^{204}\mathrm{Pb}$ *vs.* $D_{238} = {}^{206}\mathrm{Pb}/{}^{204}\mathrm{Pb}$, should produce a straight line for which the slope depends only upon the age and the (measurable) present-day relative abundance of $^{235}\mathrm{U}$ and $^{238}\mathrm{U}$.

b. The uranium isotope ratio in the Allende meteorite seems to be uniform throughout the silicate and metallic components, and has the value $N_{235}/N_{238} = {}^{235}\mathrm{U}/{}^{238}\mathrm{U} = 0.007254$ (Shimamura & Lugmair 1984), close to the ratio in Earth's crust. The relative abundances of the lead isotopes in CAIs found within Allende have been measured as follows (Connelly et al. 2008):

| Mineral sample from CAI | $^{206}\mathrm{Pb}/{}^{204}\mathrm{Pb}$ | $^{207}\mathrm{Pb}/{}^{204}\mathrm{Pb}$ |
|---|---:|---:|
| Plagioclase F2 | 42.858 | 30.783 |
| Melilite F2 | 507.1 | 321.2 |
| Pyroxene F2 | 507.1 | 321.2 |
| Plagioclase TS32 | 140.5 | 92.23 |
| Melilite TS32 | 1094.1 | 688.56 |
| Pyroxene TS32 | 421.941 | 268.139 |
| Plagioclase TS33 | 119.23 | 78.576 |
| Melilite TS33 | 378.4 | 240.9 |

| | | |
|---|---|---|
| Pyroxene TS33 | 414.77 | 263.78 |

Plot these values, fit a line to them, and thereby measure the slope. (In Excel, put them in adjacent columns and use **`Insert… Scatter…`** to generate the graph; then click on one of the plotted points to select the data and chose **`Chart Tools… Layout… Trendline…`** to generate a best-fitting line and to display its equation, which contains the slope.) Then verify that this value of slope, plus the decay rates and uranium abundance ratio above, is consistent with an age of 4567.7 Myr for the Allende CAIs.

c. **(Optional)** Either graphically, or by using numerical means, *calculate* the age of the Allende CAIs from your measured value of the slope of the $^{207}\mathrm{Pb}/^{204}\mathrm{Pb}\ -\ ^{206}\mathrm{Pb}/^{204}\mathrm{Pb}$ isochrones.

**Learn your way around the sky, lesson 7.** (An *exclusive* feature of AST 111 recitations.) Use the lab's celestial globes, TheSky running on the lab computers, and any other resources you would like to use, to answer these questions about the celestial sphere and the constellations.

4. Which way are the cardinal directions (north, south, east, west) in images as displayed by the CCD camera controller on the Mees Observatory 24-inch telescope?

5. A double star is observed at Mees. From the images it looks as if the components are displaced by 21 pixels vertically and 41 pixels horizontally. The stars are known from previous, careful observations to be 22.7 arcsec apart. So what is the pixel size on the CCD array, in arcseconds? What are the dimensions of the field of view of the camera (square, 512 pixels on a side), in arcminutes?

6. Jupiter and Callisto are observed on a night during which the two were too far apart to get in the same image. Instead an image is taken with Jupiter centered in the field, and the telescope coordinates are recorded: $\alpha = 23^{\mathrm{h}}46^{\mathrm{m}}35.0^{\mathrm{s}}$, $\delta = -03°10'21''$. Immediately thereafter an image is taken with Callisto centered in the field, and again the telescope coordinates are recorded, this time showing $\alpha = 23^{\mathrm{h}}47^{\mathrm{m}}07.8^{\mathrm{s}}$, $\delta = -03°06'42''$. How many *pixels* apart would the centers of Jupiter and Callisto be, if our CCD were big enough?

7. Closer inspection of those images later reveal that Jupiter and Callisto weren't quite centered the same, and that Callisto was in fact five pixels lower and four pixels further to the right, on the CCD camera display, than Jupiter. With these corrections, how many pixels apart would the centers of Jupiter and Callisto be?

**Problem solutions**

1. Power is generated from without by absorbing what sunlight isn't reflected, and from within by radioactivity, and is emitted from the sphere's surface:

$$P_{\rm in} = P_{\rm out}$$
$$(1-A)\frac{L}{4\pi r^2}\pi R^2 = \varepsilon\sigma T_s^4 4\pi R^2$$
$$T_s = \left(\frac{1-A}{\varepsilon}\frac{L}{16\pi\sigma r^2}\right)^{1/4}$$

2. 0 K: that is, they have exactly the same temperature. If albedo and emissivity do not depend upon wavelength, then $\varepsilon(\mathrm{IR}) = 1 - A(\text{UV-vis})$, and

$$T_s = \left(\frac{1-A}{\varepsilon}\frac{L_\odot}{16\pi\sigma r^2}\right)^{1/4} = \left(\frac{L_\odot}{16\pi\sigma r^2}\right)^{1/4} = T_{\rm blackbody}.$$

3. a. As usual,

$$N_{235}(t) = \frac{^{235}\mathrm{U}}{^{204}\mathrm{Pb}} = N_{235,0}e^{-\lambda_{235}t} \qquad N_{238}(t) = \frac{^{238}\mathrm{U}}{^{204}\mathrm{Pb}} = N_{238,0}e^{-\lambda_{238}t}$$
$$D_{235}(t) = \frac{^{207}\mathrm{Pb}}{^{204}\mathrm{Pb}} = D_{235,0} + N_{235,0} - N_{235} = D_{235,0} + N_{235}\left(e^{\lambda_{235}t} - 1\right)$$

Similarly,
$$D_{238}(t) = \frac{^{206}\mathrm{Pb}}{^{204}\mathrm{Pb}} = D_{238,0} + N_{238}\left(e^{\lambda_{238}t} - 1\right)$$

Subtract the initial values of the daughters from each side of the last two expressions and divide:

$$\frac{D_{235} - D_{235,0}}{D_{238} - D_{238,0}} = \frac{^{235}\mathrm{U}}{^{238}\mathrm{U}}\frac{e^{\lambda_{235}t} - 1}{e^{\lambda_{238}t} - 1}$$
$$D_{235} - D_{235,0} = \frac{^{235}\mathrm{U}}{^{238}\mathrm{U}}\frac{e^{\lambda_{235}t} - 1}{e^{\lambda_{238}t} - 1}\left(D_{238} - D_{238,0}\right) = m\left(D_{238} - D_{238,0}\right),$$

which is the equation of a line with slope $m = \left(^{235}U/^{238}U\right)\left(e^{\lambda_{235}t} - 1\right)\big/\left(e^{\lambda_{238}t} - 1\right)$ that passes through the point $\left(D_{238,0}, D_{235,0}\right)$.

b. Done in Excel as described, in Figure 1. I got a slope of $m$ = 0.6256. With $t$ = 4567.7 Myr, and with $^{235}\mathrm{U}/^{238}\mathrm{U} = 0.007254$ as above, we get

$$m = \frac{^{235}\mathrm{U}}{^{238}\mathrm{U}}\frac{e^{\lambda_{235}t} - 1}{e^{\lambda_{238}t} - 1} = 0.62561 \quad ,$$

which is pretty close.

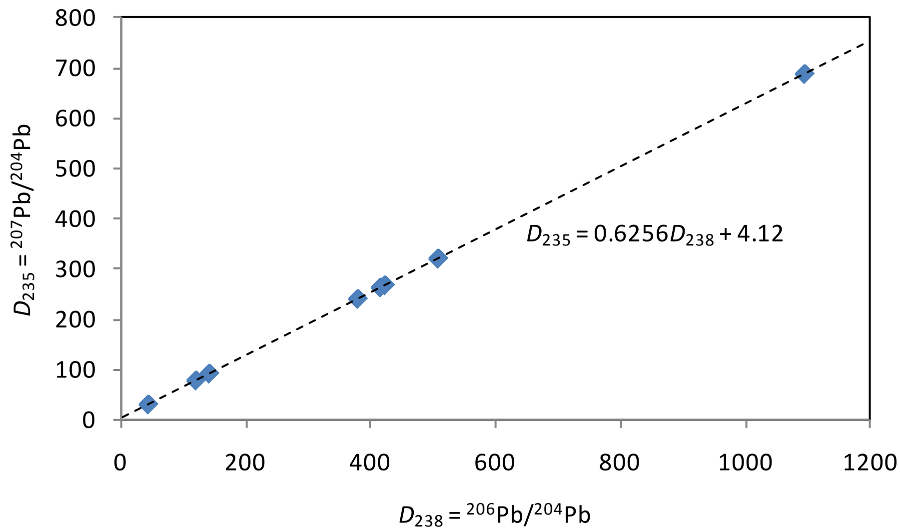


Figure 1: lead-lead isochrones for Allende CAIs, using the Connelly et al. 2008 data.

c. The slope *m*, which we have measured in Figure 1, does not lead algebraically to a value of *t*: the function $m = \left({}^{235}U/{}^{238}U\right)\left(e^{\lambda_{235}t} - 1\right)/\left(e^{\lambda_{238}t} - 1\right)$ is transcendental, and must be solved either graphically or numerically. For a graphical solution one may write

$$\frac{{}^{235}\mathrm{U}}{{}^{238}\mathrm{U}}\left(e^{\lambda_{235}t} - 1\right) = m\left(e^{\lambda_{238}t} - 1\right) \quad ,$$

and plot

$$f(t) = \frac{{}^{235}\mathrm{U}}{{}^{238}\mathrm{U}}\left(e^{\lambda_{235}t} - 1\right)$$
$$g(t) = m\left(e^{\lambda_{238}t} - 1\right)$$

as functions of *t*; the intersection of the resulting two curves gives the solution for *t*. This is done using Excel, in Figure 2. Reading off the graph I get *t* = 4567.7 Myr.

Or one can "simply" find the root of the function

$$h(t) = \frac{{}^{235}\mathrm{U}}{{}^{238}\mathrm{U}}\left(e^{\lambda_{235}t} - 1\right) - m$$

numerically; that is, find *t* such that *h*(*t*) = 0. This of course is the same as graphing *h* and finding its intersection with the *t* axis. I did it in Mathcad, using the **`root`** function, which in turn uses the Secant method for root finding. Unsurprisingly I got 4567.7 Myr. So did Connelly et al (2008) with the same data, though they plot it differently in their paper.

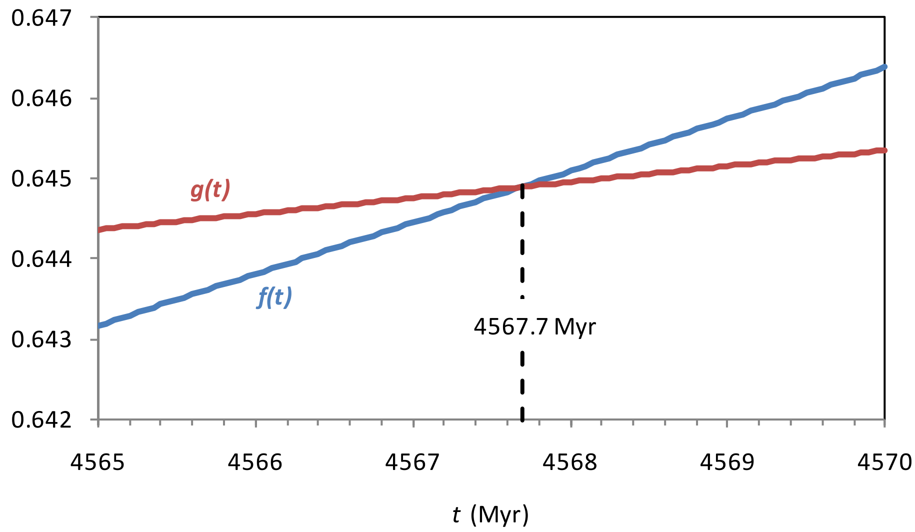


Figure 2: graphical solution for time from slope of Pb-Pb isochrones.

4. In the telescope/CCD-camera instructions we find (Figure 8 there, Figure 3 here):

| Paddle button pressed... | | | Star moves on image... | | |
|---|---|---|---|---|---|
| | N | | | ↑ | |
| W | | E | → | | ← |
| | S | | | ↓ | |

Figure 3: paddle-button response for the current camera orientation. The telescope moves in the directions indicated by the buttons, of course, so in this orientation the image has north *down* and east *right*; that is, **it's upside down compared to the way we usually display celestial images (north up, east left).**

5. The distance between the centers of the two stellar images is $\sqrt{21^2 + 41^2} = 46.1$ pixels, so the width of a pixel is 22.7″/46.1 = 0.494″. By the same token the width of the array is (512/46.1)22.7″ = 252.3″ = 4.2 arcminutes.

6. In decimal degrees, the coordinates are

$$\alpha_J = 356.6458^\circ \quad \delta_J = -3.1725^\circ$$
$$\alpha_C = 356.7825^\circ \quad \delta_C = -3.1117^\circ$$

Note that $\cos\delta = 0.9985$ for both objects, to four-place accuracy. Subtract the coordinates of the Jupiter image from those of the Callisto image:

$$\Delta\alpha\cos\delta = 491.3'' = 997.0 \text{ pixels}$$
$$\Delta\delta = 219.0'' = 444.4 \text{ pixels}$$
$$\sqrt{(\Delta\alpha\cos\delta)^2 + \Delta\delta^2} = 1091.6 \text{ pixels}$$

  

7. Since we subtracted the Jupiter-image coordinates from the Callisto-image coordinates and got positive numbers in both dimension, Callisto is west and north of Jupiter on the date in question. On the images, further down is further north, and further right is further east (see problem 4 above), so the corrections go as follows:

$$\Delta\alpha\cos\delta = 997.0 - 4 \text{ pixels}$$
$$\Delta\delta = 219.0'' = 444.4 + 5 \text{ pixels}$$
$$\sqrt{(\Delta\alpha\cos\delta)^2 + \Delta\delta^2} = 1090.0 \text{ pixels}$$

These last few problems will be good exercise for those who observed on nights in which we had to offset to find Callisto.

 