Astronomy 111 Recitation #8

27-28 October 2011

Formulas to remember

Albedo, emissivity, and solar heating of Solar-system bodies

With A as the albedo at UV and visible wavelengths, and ε the emissivity at mid-infrared wavelengths, the surface temperature of a body a distance r from the Sun is

$$T_s = \left(\frac{1-A}{\varepsilon}\frac{L_\odot}{16\pi\sigma r^2}\right)^{1/4}$$ uniform-temperature surface

$$T_s = \left(\frac{1-A}{\varepsilon}\frac{L_\odot}{16\pi\sigma r^2} + \frac{M\Lambda}{4\pi\sigma\varepsilon R^2}\right)^{1/4}$$ uniform-temperature surface, sunlight and internal radioactive heating

$$T_s = \left(\frac{1-A}{\varepsilon}\frac{L_\odot}{4\pi\sigma r^2}\cos\theta\right)^{1/4} = T_0\left(\cos\theta\right)^{1/4}$$ Slow rotator: θ = angle from subsolar point, T_0 = temperature of subsolar point, negligible internal radioactive heating.

You should be prepared to *derive* these, and similar, formulas.

Geometric albedo is the fraction of light that is reflected right back the way it came. *Bond albedo* is the fraction of light reflected and scattered in all directions, and is usually the albedo of choice for surface-temperature calculations.

Heat conduction

Heat flux: $f_T\left(r\right) = -\kappa_T\left(r\right)\frac{dT}{dr}$, where κ_T is the thermal conductivity of the material through which the heat flows.

The Poisson equation for heating and heat conduction within a spherically-symmetric body:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = -\frac{\rho\Lambda}{\kappa_T}$$

Solution to the Poisson equation for a uniform density spherical planet (radius R) and a temperature-independent thermal conductivity:

$$T\left(r\right) = T_s + \frac{\rho\Lambda}{6\kappa_T}\left(R^2 - r^2\right) \quad .$$

We'll enumerate the important results on atmospheres next week.

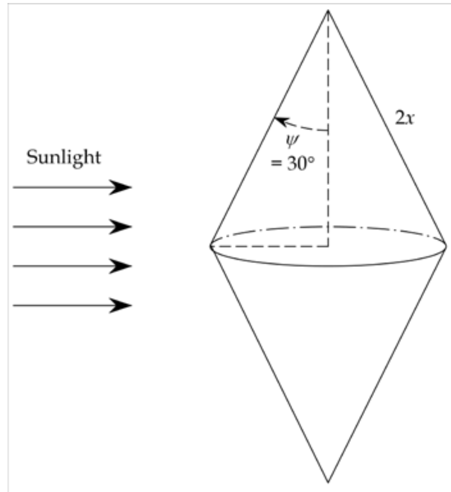
Workshop problems (do after discussing Exam #1 and Homework #6):

1. A thin, square wafer with area a has albedo A and infrared emissivity ε at all incidence angles. It lies in interplanetary space, a distance r from the Sun, and it is tilted: sunlight is incident on its surface at

angle θ; that is, the rays of sunlight make an angle θ with the direction perpendicular to the area. What is its temperature? At what angle should it be tipped so that its temperature is half the value obtained for normal incidence (i.e. perpendicular incidence, $\theta = 0$)?

2. A solid body in the shape of two identical cones joined at the base, with dimensions as shown at right, is in a circular orbit with radius r about a star with luminosity L. The axis of the cones is perpendicular to the orbit, and the body is spinning rapidly about this axis. The cone surfaces have uniform Bond albedo A_b and infrared emissivity ε. Radioactive heating is negligible.

   Derive a formula for the body's surface temperature. (*Hint*: will the temperature vary over the surface of the cones? Will the poles be colder than the equator?)



3. *Energy conservation*. Suppose that we decided to get rid of our streetlights by making the Moon a lot brighter: specifically a factor of 9 brighter, by covering the near side of the moon with a shiny surface that has uniform Bond albedo 0.99. Assume that this results in the near side of the Moon reflecting sunlight *uniformly* into the hemisphere of the sky above the illuminated part of the Moon. What would be the flux (in erg $\sec^{-1}$ cm^{-2}) of reflected sunlight at the Earth's surface when the Moon is full and directly overhead? Compare this with the flux from sunlight at noon, with Sun directly overhead.

4. In Cartesian coordinates and one dimension, the Poisson equation for heat conduction looks like this:

$$\frac{d^2T}{dz^2} = -\frac{\rho\Lambda}{\kappa_T}$$

   An infinite slab of solid material has its planar surfaces lying at $z = 0$ and $z = H$, and these surfaces are held at constant and uniform temperatures T_0 and T_H, respectively. There are no radioactive elements in sight. Derive a formula for the temperature as a function of z within the slab.

5. Repeat the derivation done in lecture Thursday of the internal temperature of a uniform-density planet,

$$T(r) = T_s + \frac{\rho\Lambda}{6\kappa_T}\left(R^2 - r^2\right) \quad .$$

   but using definite integrals instead of indefinite integrals and integration constants. (*Hint:* consider integrating from 0 to r, and then from r to R.)

6. Consider again the simple differentiated planet from recitation on 22-23 September 2011: it has mass M, radius R, a uniform-density core with density ρ_c and radius $R_c = 2R/3$, and a uniform-density mantle and crust with density $\rho_m = 2\rho_c/5$ for $r = R_c$ to R. You showed, during that recitation, that the mass M and core density ρ_c are related by $\rho_c = 135M/104\pi R^3$, and that its moment of inertia is $I = 0.33MR^2$, so that its internal structure may be much like Earth's. In Homework #6 you will show

that the temperature within the planet's mantle, with thermal conductivity κ_m, and radioactive heating power per unit mass Λ, is

$$T(r) = T_s + \frac{\Lambda}{6\kappa_m}\left[\rho_m\left(R^2 - r^2\right) + \frac{2R_c^3}{3}\left(\rho_c - \rho_m\right)\frac{R-r}{Rr}\right] \quad , R_c < r \le R.$$

a. The core has thermal conductivity κ_c, and has radioactive heating power per unit mass Λ the same as the mantle. Show that the temperature within the core is given by

$$T(r) = T_s + \frac{\Lambda}{6\kappa_m}\left[\rho_m\left(R^2 - R_c^2\right) + \frac{2R_c^3}{3}\left(\rho_c - \rho_m\right)\frac{R-R_c}{RR_c}\right] + \frac{\rho_c\Lambda}{6\kappa_c}\left(R_c^2 - r^2\right) \quad , r \le R_c.$$

b. *"The interior of the Earth is extremely hot, several million degrees…"*

-- Al Gore, 12 November 2009, on Conan O'Brien.

Suppose the core is liquid nickel-iron and the mantle is silicate rock; reasonable parameters for them are

$\Lambda = 5.52\times10^{-8}$ erg sec^{-1} gm^{-1}

$\kappa_c = 9.10\times10^{6}$ erg sec^{-1} cm^{-1} K^{-1} (liquid iron at 4500 K)

$\kappa_m = 6.38\times10^{5}$ erg sec^{-1} cm^{-1} K^{-1} (solid rock at 3000 K)

Calculate the temperature at the center of the planet. Compare this result with Mr. Gore's statement, and the result obtained by seismology on Earth, T = 5000 K.

c. That didn't work so well; instead let's use $\kappa_m = 3.33\times10^{6}$ erg sec^{-1} cm^{-1} K^{-1}, with all other parameters the same, and calculate the temperature at the center of the planet again. What do we get this time?

d. So $T(r)$ for this model is pretty close to the real results for Earth if we increase the thermal conductivity of the mantle by a factor of about 5 over that of real rocks at the mantle's temperature. What does that suggest about the dominant mode of heat transfer in the mantle?

Learn your way around the sky, lesson 8. (An *exclusive* feature of AST 111 recitations.) Use the lab's celestial globes, TheSky running on the lab computers, and any other resources you would like to use, to answer these questions about the celestial sphere, the constellations, and in this case the influence of Earth's orbital motion.

7. Some new terminology about planetary orbits:

   Opposition means that Earth, planet and Sun lie along a line, with the Sun on one end. **Conjunction** means that they lie along a line with the Sun in the middle. The **synodic period** of a planet other than Earth is the time interval between consecutive oppositions or conjuctions of Earth with that planet. A **inferior planet** is follows an orbit contained within Earth's; Earth's orbit is contained within the orbit of a **superior planet.**

a. Show that the synodic period Π and sidereal period P of an inferior planet (Mercury or Venus) are related by $\frac{1}{\Pi} = \frac{1}{P} - \frac{1}{P_\oplus}$, and those of a superior planet (Mars – Neptune) by $\frac{1}{\Pi} = \frac{1}{P_\oplus} - \frac{1}{P}$, where $P_\oplus$ is one sidereal year on Earth. (*Hint:* start by thinking of angular frequency instead of period, and find out how long it takes the faster of two bodies travelling at different angular speeds to cover one more orbit than the slower one.)

b. The last time we had a really close Martian opposition was 27 August 2003 (see problem set 4, problem number 5). How many oppositions have we had in the meantime?

c. Look it up: when is the next opposition of Jupiter?

Problem solutions

1. The area of the square's shadow is $a\cos\theta$:

$$
\begin{aligned}
P_{\text{in}} &= P_{\text{out}} \\
\frac{L_\odot}{4\pi r^2}(1-A)\,a\cos\theta &= \varepsilon\sigma T^4\, 2a \\
T &= \left(\frac{1-A}{\varepsilon}\frac{L_\odot\cos\theta}{8\pi\sigma r^2}\right)^{1/4}
\end{aligned}
$$

The maximum T is obtained for $\cos\theta = 1$ $(\theta = 0)$, so the angle for half the maximum is determined by

$$
\begin{aligned}
T = \left(\frac{1-A}{\varepsilon}\frac{L_\odot\cos\theta}{8\pi\sigma r^2}\right)^{1/4} &= \frac{1}{2}T_{\max} = \frac{1}{2}\left(\frac{1-A}{\varepsilon}\frac{L_\odot}{8\pi\sigma r^2}\right)^{1/4} \\
\frac{1-A}{\varepsilon}\frac{L_\odot\cos\theta}{8\pi\sigma r^2} &= \frac{1}{16}\frac{1-A}{\varepsilon}\frac{L_\odot}{8\pi\sigma r^2} \\
\cos\theta = \frac{1}{16} \quad &\Rightarrow \quad \theta = 1.51\ \text{rad} = 86^\circ.
\end{aligned}
$$

2. The area of the bi-cone's shadow is the same as its bi-triangular cross section. Note that each of the triangles is equilateral, and recall that a triangle's area is half its base times its height; the shadow area becomes

$$
S_\perp = 2\times\frac{2x\times 2x\cos 30^\circ}{2} = 2\sqrt{3}x^2 \quad .
$$

Then recall that the surface area of a cone, without its circular base, is π times its base radius times its hypotenuse:

$$
S = 2\times\pi\times x\times 2x = 4\pi x^2 \quad .
$$

Now we can proceed as usual:

$$
\begin{aligned}
P_{\text{in}} &= P_{\text{out}} \\
\frac{(1-A_b)L}{4\pi r^2}S_\perp &= \varepsilon\sigma T_s^4 S \\
\frac{(1-A_b)L}{4\pi r^2}2\sqrt{3}x^2 &= \varepsilon\sigma T_s^4\, 4\pi x^2 \\
T_s &= \left(\frac{1-A_b}{\varepsilon}\frac{L\sqrt{3}}{8\pi^2\sigma r^2}\right)^{1/4} \quad .
\end{aligned}
$$

3. The power in sunlight incident on the Moon (distance r from the Sun) is

$$
P_{\text{in},\,M} = \frac{L_\odot}{4\pi r^2}\pi R_M^2 \quad ,
$$

of which a fraction $A_{b,M}$ reflects, and is scattered uniformly into half of the sky. By the time it gets to Earth (distance r_M from the Moon) it is uniformly spread over a hemisphere with radius r_M, so the flux at Earth is

$$f_{M-\oplus} = \frac{A_{b,M} P_{\text{in},M}}{2\pi r_M^2} = \frac{A_{b,M} L_\odot R_M^2}{8\pi r^2 r_M^2} = 13.7 \text{ erg sec}^{-1} \text{ cm}^{-2} \quad .$$

This is a lot – it would be like having every point on the surface lie within 2400 cm (80 feet) of a 100-watt light bulb, and that's a lot more coverage than we get from current streetlights. (Though this won't work so well at new moon!) But it's still vastly less than the flux in sunlight at Earth's subsolar point:

$$f_{S-\oplus} = \frac{L_\odot}{4\pi \text{AU}^2} = 13.6\times 10^5 \text{ erg sec}^{-1} \text{ cm}^{-2} \quad ,$$

which is more like having all the points on the surface lie within 7.6 cm of a 100-watt light bulb.

4. So this is an opportunity to solve the Poisson equation with zero internal heating (which makes it the Laplace equation) in one dimension. Integrating twice over z, and calling the integration constants C and D, we have

$$\begin{aligned} \frac{d^2T}{dz^2} &= 0 \\ \frac{dT}{dz} &= C \\ T(z) &= Cz + D \quad . \end{aligned}$$

Now apply the boundary conditions:

$$\begin{aligned} T(0) &= D = T_0 \\ T(H) &= CH + T_0 = T_H \quad \Rightarrow \quad C = \frac{T_H - T_0}{H} \quad . \end{aligned}$$

Thus,

$$T(z) = \frac{T_H - T_0}{H} z + T_0 \quad .$$

5. To make it clearer change variables slightly in the Poisson equation:

$$\frac{1}{r''^2}\frac{d}{dr''}\left(r''^2\frac{dT}{dr''}\right)=-\frac{\rho\Lambda}{\kappa_T}$$

$$\int_0^{r'}\frac{d}{dr''}\left(r''^2\frac{dT}{dr''}\right)dr=-\frac{\rho\Lambda}{\kappa_T}\int_0^{r'}r''^2dr''$$

$$r''^2\frac{dT}{dr''}\bigg|_0^{r'}=-\frac{\rho\Lambda}{3\kappa_T}r''^3\bigg|_0^{r'}$$

$$\frac{dT}{dr'}=-\frac{\rho\Lambda}{3\kappa_T}r'$$

$$\int_r^R\frac{dT}{dr'}dr'=-\frac{\rho\Lambda}{3\kappa_T}\int_r^R r'dr' \quad \text{Note that } T(R)=T_s:$$

$$T_s-T(r)=-\frac{\rho\Lambda}{6\kappa_T}\left(R^2-r^2\right)$$

$$T=T_s+\frac{\rho\Lambda}{6\kappa_T}\left(R^2-r^2\right)$$

6. a. If one integrates the Poisson equation for heat transfer, one gets

$$T(r)=-\frac{\rho_c\Lambda}{6\kappa_c}r^2-\frac{C}{r}+D$$

for the core. As usual the temperature has to be finite at the center, and it would blow up unless $C = 0$. The temperature has to match up with the mantle at $r=R_c$, so

$$T(R_c)=-\frac{\rho_c\Lambda}{6\kappa_c}R_c^2+D=T_{\text{mantle}}(R_c)$$
$$=T_s+\frac{\Lambda}{6\kappa_m}\left[\rho_m\left(R^2-R_c^2\right)+\frac{2R_c^3}{3}(\rho_c-\rho_m)\frac{R-R_c}{RR_c}\right]$$

This determines the other integration constant:

$$D=T_s+\frac{\Lambda}{6\kappa_m}\left[\rho_m\left(R^2-R_c^2\right)+\frac{2R_c^3}{3}(\rho_c-\rho_m)\frac{R-R_c}{RR_c}\right]+\frac{\rho_c\Lambda}{6\kappa_c}R_c^2 \quad ,$$

and thus the core temperature is as advertised,

$$T(r)=T_s+\frac{\Lambda}{6\kappa_m}\left[\rho_m\left(R^2-R_c^2\right)+\frac{2R_c^3}{3}(\rho_c-\rho_m)\frac{R-R_c}{RR_c}\right]+\frac{\rho_c\Lambda}{6\kappa_c}\left(R_c^2-r^2\right) \quad .$$

b. I get T = 17,700 K, a lot larger than the measured temperature at Earth's center. The whole temperature profile is plotted in Figure 1.

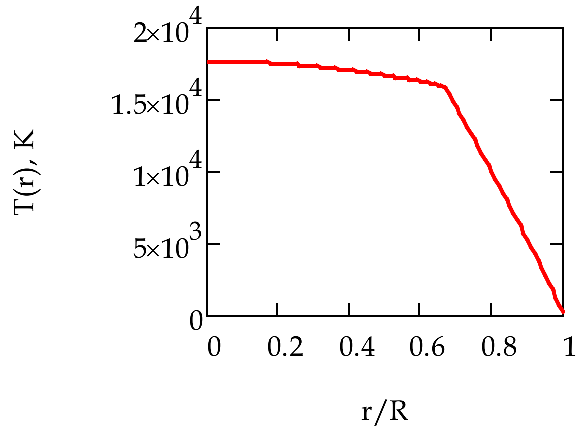


Figure 1: temperature profile of the simple differentiated planet, with material parameters as in part b. Note how uniform in temperature the core is, a result of its high thermal conductivity.

Mr. Gore, of course, was a C student at Harvard. Not that I think he's evil or anything like that, but I'm glad the party has switched back to A students.

c. This time I get T = 5000 K on the dot. (Funny how that works.) In Figure 2 is shown the resulting temperature profile. Go on line and compare this to the seismographically-measured interior temperature of the Earth, and you'll see it's pretty similar.

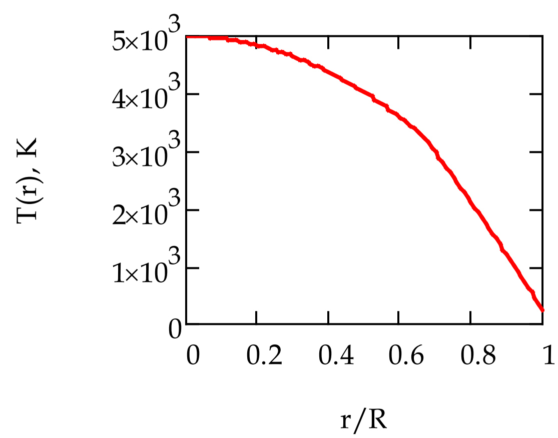


Figure 2: temperature profile of the simple differentiated planet, with material parameters as in part c. Increasing the thermal conductivity of the mantle has dropped the core temperature substantially; the whole curve is in pretty good agreement with measurements of Earth's interior temperature, except for the position of the slight kink; the real Earth core is a bit smaller than we've assumed here.

d. Conduction apparently does not dominate heat transfer in the mantle. The next contender is convection, which would be reasonably efficient even if the mantle is only plastic, rather than properly liquid.

5. a. Consider two planets, the one closer to the Sun (inferior) travelling at sidereal angular velocity ω_I and the outer one at angular velocity ω_S. In one synodic period *S*, the outer planet has covered an angle $\omega_S S$ in its orbit, but the faster, inner planet has lapped the outer one: it has covered that same angle $\omega_S S$ plus a complete orbit. Thus

$$\omega_I S = \omega_S S + 2\pi \quad .$$

Since the period and angular velocity (for circular orbits) are related by $\omega = 2\pi/P$, this is the same as

$$\frac{1}{P_I} = \frac{1}{P_S} + \frac{1}{S} \quad .$$

For Mercury and Venus, Earth is superior; it is inferior to all the others. Thus

$$\begin{aligned} \frac{1}{S} &= \frac{1}{P_I} - \frac{1}{P_S} = \frac{1}{P} - \frac{1}{P_\oplus} \qquad \text{Mercury, Venus} \quad , \\ &= \frac{1}{P_\oplus} - \frac{1}{P} \qquad \text{Mars - Neptune} \quad . \end{aligned}$$

b. The synodic period of Mars is

$$S = \frac{P_\oplus P}{P - P_\oplus} = \frac{1.881}{0.881} \text{ years} = 2.135 \text{ years} = 779.8 \text{ days.}$$

Since that close opposition, eight years and 61 days (2983 days) have passed. Thus there have been three oppositions since then. The next one is on 3 March 2012. This one will be a fairly *distant* opposition, as Mars will be near aphelion at the time.

c. This Saturday. (29 October 2011.)

