Astronomy 111 Recitation #11

17-18 November 2011

Formulas to remember

Photon energy and momentum

$$E = \frac{hc}{\lambda} \qquad p = \frac{h}{\lambda} = \frac{E}{c}$$

Radiation force of starlight on a perfect absorber (blackbody)

$$F_{\mathrm{rad}} = \frac{dp_{\mathrm{in}}}{dt} = \frac{1}{c}\frac{dE_{\mathrm{in}}}{dt} = \frac{LA_{\mathrm{shadow}}}{4\pi c r^2} = \frac{LR^2}{4cr^2} \quad \text{if spherical with radius } R\,.$$

where L is the star's luminosity and r its distance from the absorber. For a spherical, perfect absorber in orbit around a star,

$$\boldsymbol{F} = \boldsymbol{F}_{\mathrm{rad}} + \boldsymbol{F}_{\mathrm{P\text{-}R}} = \frac{LR^2}{4cr^2}\left(\hat{\boldsymbol{r}} - \hat{\boldsymbol{\eta}}\frac{v}{c}\right) = \beta\frac{GMm}{r^2}\left(\hat{\boldsymbol{r}} - \hat{\boldsymbol{\eta}}\frac{v}{c}\right) \quad ,$$

where the two force components correspond to radiation pressure and Poynting-Robertson drag, where M and m are the masses of star and absorber, respectively, and where

$$\beta \equiv \left|\frac{F_{\mathrm{rad}}}{F_{\mathrm{gravity}}}\right| = \frac{3L}{16\pi c G M R \rho} \cong 0.96\left(\frac{0.1\,\mu\mathrm{m}}{R}\right)\left(\frac{Q}{0.5}\right)\left(\frac{3\text{ gm cm}^{-3}}{\rho}\right) \quad .$$

Time scale for Poynting-Robertson drag to pull a body into the star:

$$\tau_{\mathrm{P\text{-}R}} = \frac{cr^2}{\beta G M} = 1600\text{ yr}\left(\frac{r}{\mathrm{AU}}\right)^2\frac{1}{\beta} \quad .$$

Workshop problems (do after remaining discussion of Homework #8):

Warning! The workshop problems you will do in groups in Recitation are a crucial part of the process of building up your command of the concepts important in AST 111 and subsequent courses. Do not, therefore, do your work on scratch paper and discard it. Better for each of you to keep your own account of each problem, in some sort of bound notebook.

1. Absorption of a photon can be viewed as a completely inelastic collision; reflection, as an elastic one.

   a. In one dimension, write down the initial and final momentum, p_i and p_f, involved in a collision between a packet of N photons with energy E, and a flat blackbody with mass m, initially at rest and perpendicular to the incoming light. (Note that photons are massless!)

   b. Take the derivative with respect to time of this expression. Identify parts of the result as the input power P_{in} and the force F_{rad} on the blackbody, to show that

$$\frac{dp_i}{dt} = \frac{1}{c}P_{\text{in}} = \frac{dp_f}{dt} = F_{\text{rad}} \quad .$$

c. Suppose that the photons come from a star with luminosity *L*, a distance *r* away, and that the blackbody has an area *A*. Derive the expression for radiation force obtained in class on Tuesday, from the result of part b.

2. a. Now start over: in one dimension, write down the initial and final momentum, p_i and p_f, involved in a collision between a packet of *N* photons with energy *E,* and a flat *mirror* with mass *m,* initially at rest and perpendicular to the incoming light.

   b. Again, take the derivative of this expression, identify the parts of the expression as before, and show that you get

$$\frac{2}{c}P_{\text{in}} = F_{\text{rad}}$$

   -- that is, that the same incident power exerts twice as large a radiation force on a mirror as it does on a perfect absorber.

3. a. Start over yet again: suppose that this time we have a flat, thin wafer oriented with one side perpendicular to incoming light. Both faces have albedo A_b and emissivity ε. Write down the initial and final momentum, p_i and p_f, involved in a collision between a packet of *N* photons with energy *E,* and this wafer.

   b. Again, take the derivative of this expression, identify the parts of the expression as before, and show that you get

$$\frac{1+A_b}{c}P_{\text{in}} = F_{\text{rad}}$$

   c. Again, suppose that the photons come from a star with luminosity *L*, a distance *r* away, and that the blackbody has an area *A*. Derive the expression for radiation force, and compare it to that implied by the value of β claimed in Thursday's lecture.

4. (Somewhat tricky.) In a perfectly elastic collision, kinetic energy is conserved. Consider kinetic energy balance in problem 2 above. Any guesses as to the origin of the kinetic energy of the mirror? (That is, what small effect might we have left out in this solution, which would make the kinetic energy balance?)

Learn your way around the sky, lesson 11. (An *exclusive* feature of AST 111 recitations.) Use the lab's celestial globes, TheSky running on the lab computers, SIMBAD at http://simbad.harvard.edu/simbad/sim-fid, and any other resources you would like to use, to answer these questions about the celestial sphere, the constellations, and the orbits of the planets.

5. As the Winter Triangle rises, and the Summer Triangle sets, a few other distinctive objects are high in the sky. Identify and describe these three objects, which lie roughly at the vertices of a big 30-60-90 triangle on the sky:

   a. the constellation Cassiopeia.

   b. Capella.

   c. the Pleiades.

6. Suppose I know how to find those three landmarks. Describe how I find

   a. the constellation Perseus.

   b. the constellation Andromeda.

   c. the constellation Cepheus.

Problem solutions

1. a. $p_i = N\frac{E}{c} = p_f = mv.$

b. $$\frac{dp_i}{dt} = \frac{E}{c}\frac{dN}{dt} = \frac{dp_f}{dt} = ma$$
$$\frac{1}{c}\frac{d}{dt}(NE) = \frac{1}{c}P_{\text{in}} = ma = F_{\text{rad}}$$

c. $$\frac{E}{c}\frac{dN}{dt} = \frac{1}{c}P_{\text{in}} = \frac{1}{c}\frac{LA}{4\pi r^2} = ma = F_{\text{rad}}$$

2. a. $p_i = N\frac{E}{c} = p_f = -N\frac{E}{c} + mv.$

b. $$\frac{dp_i}{dt} = \frac{E}{c}\frac{dN}{dt} = \frac{dp_f}{dt} = -\frac{E}{c}\frac{dN}{dt} + ma$$
$$2\frac{1}{c}\frac{d}{dt}(NE) = \frac{2}{c}P_{\text{in}} = ma = F_{\text{rad}}$$

3 a. $p_i = N\frac{E}{c} = p_f = -A_b N\frac{E}{c} + mv.$

b. $$\frac{dp_i}{dt} = \frac{E}{c}\frac{dN}{dt} = \frac{dp_f}{dt} = -A\frac{E}{c}\frac{dN}{dt} + ma$$
$$(1 + A_b)\frac{1}{c}\frac{d}{dt}(NE) = \frac{1 + A_b}{c}P_{\text{in}} = ma = F_{\text{rad}}$$

c. $$F_{\text{rad}} = \frac{1 + A_b}{c}P_{\text{in}} = \frac{1 + A_b}{c}\frac{LA}{4\pi r^2}$$

So, $\beta = \left|\frac{F_{\text{rad}}}{F_{\text{grav}}}\right| = \frac{1 + A_b}{c}\frac{LA}{4\pi r^2}\frac{r^2}{GMm} = \frac{(1 + A_b)L}{4\pi cGM}\frac{A}{m}$, which is the same as claimed on Tuesday apart from the fact that on Tuesday we were talking about spheres instead of wafers, and allowing for the possibility of scattering $(Q \neq 1)$.

Avoid confusion: in the rather similar looking expressions for temperature of a starlight-heated, emitting body, the quantity $1 - A_b$ appears, and here it's $1 + A_b$.

4. If the photons reflect each with the same energy they had before encountering the mirror, their (kinetic) energy would be the same as they had before. Clearly something is different about the reflected photons. It can't be N (we assumed a perfect mirror, so all the photons get reflected), so it must be E: each photon is reflected with a slightly smaller energy E' than it had before – or, equivalently, with a slightly longer wavelength than before. Here's how it works out, for one photon and the reflector initially at rest:

$$p_i = \frac{E}{c} = p_f = -\frac{E'}{c} + mv \quad \Rightarrow \quad E = mvc - E'$$
$$E_i = E = E_f = E' + \frac{1}{2}mv^2$$

Two equations in the unknowns E' and v. We're after E'. But we can eliminate this and solve for v just by adding these last two results:

$$2E = mvc + \frac{1}{2}mv^2$$
$$v^2 + 2cv - \frac{4E}{m} = 0$$
$$v = \frac{1}{2}\left(-2c \pm \sqrt{4c^2 + 4\frac{4E}{m}}\right)$$

Choose the positive root and substitute back into the first expression above (momentum conservation):

$$v = c\sqrt{1 + \frac{4E}{mc^2}} - c$$
$$E' = mvc - E = mc^2\sqrt{1 + \frac{4E}{mc^2}} - mc^2 - E$$

This exact expression would be good enough as an answer. If you put numbers into this expression you will see that E' comes out less than E. Very slightly less, though. In fact if one uses our first order approximation, which should be very good because E should be tiny compared to mc^2,

$$(1+x)^n \cong 1 + nx \quad ,$$

one finds

$$E' = mc^2\sqrt{1 + \frac{4E}{mc^2}} - mc^2 - E$$
$$\cong mc^2\left(1 + \frac{2E}{mc^2}\right) - mc^2 - E = E \quad \text{(first-order)}$$

This of course is why it was OK to use $E' = E$ to good approximation in problems 1-3. So to put the result in a form suggestive enough that we won't have to do arithmetic to verify that $E' < E$, I'll introduce here the *second* order approximation:

$$(1+x)^n = \sum_{i=0}^{\infty} \frac{n!}{i!(n-i)!}x^i \cong 1 + nx + \frac{n(n-1)}{2}x^2 \quad .$$

With this,

$$
\begin{aligned}
E' &= mc^2\sqrt{1+\frac{4E}{mc^2}} - mc^2 - E \\
&\cong mc^2\left(1+\frac{2E}{mc^2}-\frac{1}{8}\left[\frac{4E}{mc^2}\right]^2\right) - mc^2 - E \\
&= E - \frac{2E^2}{mc^2} = E\left(1-\frac{2E}{mc^2}\right) < E \quad \text{, obviously.}
\end{aligned}
$$

5. a. Cassiopeia is a W-shaped constellation with the top of the W facing the north pole; it is nearly overhead in the early evening these days, as the Winter Triangle rises and the Summer one sets.

   b. Capella (α Aurigae) is the brightest star in the sky north of the Winter Triangle. You can probably find it at a glance, but to make sure: go halfway along the line between Betelgeuse and Gemini (see last week's Recitation), and then upwards, toward the west, to the bright orange star.

   c. The Pleiades is a fuzzy patch of nebulosity surrounding a cluster of faint stars, that is as distinctive a landmark as anything in the sky. Find it from the Winter Triangle by starting at the midpoint of Sirius-Procyon, heading off in the direction of Betelgeuse, and going the same distance past Betelgeuse along the same line. But it's easier to find it in a glance than to follow directions.

6. a. Perseus is right in the center of the Cassiopeia-Capella-Pleiades triangle. Look for a smaller, acute triangle which includes the brightest Perseus stars: Mirfak, Algol and δ Persei.

   b. Andromeda is nested between Cassiopeia and Perseus. Its three brightest stars are stretched out on a long line from Capella through the center of Perseus's Mirfak-Algol-δ Persei triangle. About halfway between the third of these stars (α Andromedae = Alpheratz) and the "left" part of Cassiopeia's W is Andromeda's most famous feature, the Great Andromeda Nebula (M31) which can be seen with the naked eye on dark nights.

   c. For Cepheus, start at the Perseus triangle and head about the same distance on the other side of Cassiopeia. Note that you will be moving along the Milky Way as you do. Cepheus is an unprepossessing constellation but lots of big star formation regions lie within its bounds because of its embrace of a section of the Galactic plane – as is also the case for Cassiopeia and Perseus.

Remind me sometime to tell you the story of Perseus, Andromeda, Cassiopeia and Cepheus.