Astronomy 111 Recitation #12

1-2 December 2011

Formulas to remember

Initial protoplanetary disk size, from angular momentum conservation

$$r = \frac{R^4\Omega^2}{GM} = \frac{R^2V^2}{GM} \sim 500\ \text{AU} \quad ,$$

where the mass of the collapsed molecular-cloud fragment is M, its initial radius is R, and the typical speed associated with the "tumbling" of molecular fragments is V.

Core-accretion growth of planetesimals

If the typical relative speed of small particles in a disk is v, and the density of such particles in the disk is ρ, then the mass flux on any solid particle is

$$f_m = \rho v \quad ..$$

The rate at which a spherical particle (mass M, radius R, density ρ_p) grows in collisions is

$$\frac{dM}{dt} = \rho v \pi R^2\left(1+\frac{v_e^2}{v^2}\right) \qquad \frac{dR}{dt} = \frac{\rho v}{4\rho_p}\left(1+\frac{v_e^2}{v^2}\right) \quad ,$$

where $v_e = \sqrt{2GM/R}$ is the particle's escape speed. Approximations for small and large particles:

$$v_e \ll v \quad \frac{dR}{dt} = \frac{\rho v}{4\rho_p} \quad \text{(constant)}$$

$$v_e \gg v \quad \begin{cases} \dfrac{dR}{dt} = \dfrac{2\pi G\rho}{3v}R^2 \\ R(t) = \dfrac{R_0}{1-\dfrac{2\pi G\rho R_0}{3v}t} \to \infty \text{ at } t = \dfrac{3v}{2\pi G\rho R_0} \end{cases} .$$

Reminder: hydrostatic equilibrium, 1-D

$$\frac{dP}{dz} = -\rho g_z$$

Gas/small-dust-grain orbital speed and large-particle headwind speed in the Solar nebula

$$v_g = v_K - \frac{13}{14}\frac{v_0^2}{v_K} \quad \text{and} \quad v_{HW} = v_K - v_g = \frac{13}{14}\frac{v_0^2}{v_K}\text{, where}$$

$$v_0 = \sqrt{2kT/\mu} \quad \text{Thermal speed of molecules in the gas}$$

$$v_K = \sqrt{GM_\odot/r} \quad \text{Keplerian orbital speed}$$

Simple model Solar nebula

Mass of star in center = $1M_\odot$, mass of disk $0.1M_\odot$.

Parameter	**Formula**	**where…**
Mass per unit disk area (gas and dust, well mixed):	$\Sigma(r) = \Sigma_0 \left(\frac{r_0}{r}\right)^2$	$\Sigma_0 = 6.74\times10^7$ gm cm^{-2}, $r_0 = 0.0142$ AU
Temperature	$T(r) = T_0 \left(\frac{r_0}{r}\right)^{3/7}$	$T_0 = 1500$ K
Gas pressure	$P(r) = P_0 \left(\frac{r_0}{r}\right)^{26/7}$	$P_0 = 7.01\times10^8$ dyne cm^{-2}
Isothermal gas-pressure scale height	$H(r) = H_0 \left(\frac{r}{r_0}\right)^{9/7}$	$H_0 = 2.98\times10^9$ cm
Total density (gas plus dust)	$\rho_t(r) = \rho_{t0} \left(\frac{r_0}{r}\right)^{23/7}$	$\rho_{t0} = 1.13\times10^{-2}$ gm cm^{-3}
Dust density, newborn disk	$\rho(r) = \xi\rho_t(r)$	$\xi = 0.01$
Dust density, sedimented disk	$\rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^{23/7}$	$\rho_0 = 5.67\times10^{-2}$ gm cm^{-3}
Dust scale height, sedimented disk	$H_d(r) = H_0 \left(\frac{r}{r_0}\right)^{9/7}$	$H_0 = 5.95\times10^6$ cm

Workshop problems (do after discussing Homework #9):

Warning! The workshop problems you will do in groups in Recitation are a crucial part of the process of building up your command of the concepts important in AST 111 and subsequent courses. Do not, therefore, do your work on scratch paper and discard it. Better for each of you to keep your own account of each problem, in some sort of bound notebook.

1. What is the radius of a $\rho_p = 1$ gm cm^{-3} spherical planetesimal for which the escape speed is equal to the headwind speed?

2. The Moon is yanked out of the solar system and placed in orbit at 1 AU about a $1M_\odot$ star, within a protoplanetary disk that just like the sedimented version of the model Solar nebula discussed in class. Is gravitational focusing important in its subsequent growth? If so, in how many years will it achieve runaway accretion? In how many years will it have twice the radius of the Earth? (Take the mean molecular mass of the gas in the disk to be the mass of molecular hydrogen.)

3. *Why disks are flared.* Consider a parcel of gas in a disk, a distance *r* away from the central star of mass *M*, and a distance *z* above midplane. The disk is *vertically isothermal*: that is, the temperature is the same for all *z* at a given *r*. It is supported in the vertical direction by gas pressure at its local temperature: that is, hydrostatic equilibrium consists of the balance between the pressure gradient and the vertical component of gravity.

a. Derive an expression for the component of the gravitational acceleration perpendicular to the disk at the location of our gas parcel, g_z, neglecting gravitation from everything but the star in the center of the disk. Your answer should depend upon *z*, *r*, and *M*. Feel free to neglect terms depending only on z in sums with terms depending only on *r*.

b. Insert this expression into the equation of hydrostatic equilibrium, and then use the ideal gas law to eliminate the density ρ in favor of the pressure *P*, and thus obtain a differential equation for *P*.

c. Separate and integrate this expression, from *z* = 0 and $P = P_0$ to *z* and *P*, and show that the result has the form

$$P(z) = P_0 e^{-z^2/H^2} \quad .$$

Give the corresponding for equation for *H*.

d. As we have shown many times, the temperature of solid bodies heated by starlight varies with radius according to the form $T = T_0 \left(r/r_0\right)^{-1/2}$. So how does *H* depend upon *r*? Compare your result to the expression for *H* we use in the simple model of the pre-Solar nebula which was introduced in lecture, $H = H_0 \left(r/r_0\right)^{9/7}$. I assert that this form is in fact consistent with hydrostatic equilibrium and starlight heating, and the form you just obtained is not. Take a guess: what did we do wrong, in this problem?

Learn your way around the sky, lesson 12. (An *exclusive* feature of AST 111 recitations.) Use the lab's celestial globes, TheSky running on the lab computers, SIMBAD at http://simbad.harvard.edu/simbad/sim-fid, and any other resources you would like to use, to answer these questions about the celestial sphere, the constellations, and the orbits of the planets.

4. In the center of the constellation Perseus is a prominent acute triangle formed by Mirfak, Algol and δ Persei. Algol occupies the vertex at the sharper tip. It has long been a favorite naked-eye object because it is an eclipsing binary – the stellar analog of the transiting exoplanetary systems we've been discussing lately – and the dimming of the system during eclipse, in comparison to nearby Mirfak and δ Persei, is dramatic enough to be easily noticeable by eye.

   a. Find out when the next eclipse of Algol is, and how long it lasts.

   b. Find the mythological connection of Algol to the story of Perseus and Andromeda.

Problem solutions

1. $v_e^2 = v_{HW}^2$ amounts to

$$v_e^2 = \frac{2G}{R}\frac{4\pi}{3}\rho_p R^3 = v_{HW}^2 = \left(\frac{13}{14}\right)^2 \frac{4k^2T_0^2}{\mu^2}\frac{r_0}{r}\frac{r}{GM_\odot}, \text{ so}$$

$$R = \left(\frac{13}{14}\right)\frac{kT_0}{G\mu}\sqrt{\frac{3r_0}{2\pi M_\odot \rho_p}} = 2.1\times 10^7 \text{ cm} = 0.03 R_\oplus \quad .$$

2. The radius of the Moon is a good deal larger than the radius just calculated in problem 1, so it will be growing at a greater than constant rate (and subject to a headwind), and will hit runaway accretion at

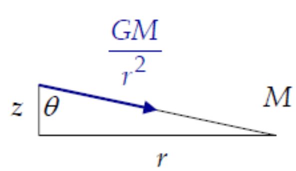
$$t(\text{runaway}) = \frac{3v_{HW}}{2\pi G\rho R_M} = \frac{3}{2\pi G R_M}\frac{13}{14}\frac{2kT_0}{\mu}\sqrt{\frac{r_0}{r}}\sqrt{\frac{r}{GM_\odot}}\frac{1}{\rho_0}\left(\frac{r}{r_0}\right)^{\frac{5}{2}}$$
$$= 7.4\times 10^{12} \text{ sec} = 2.4\times 10^5 \text{ years}$$

For the rest, solve the $R(t)$ expression for *t* and calculate, noting that most of the arithmetic from the first part of the problem can be reproduced:

$$t = \frac{3v_{HW}}{2\pi G\rho}\left(\frac{1}{R_M} - \frac{1}{2R_\oplus}\right) = 6.4\times 10^{12} \text{ sec} = 2.0\times 10^5 \text{ years.}$$

Perilously close to runaway accretion…

3. The geometry:



a. $g_z = \frac{GM}{r^2}\cos\theta = \frac{GM}{r^2}\frac{z}{\sqrt{z^2 + r^2}} \cong \frac{GM}{r^2}\frac{z}{r}$.

b. $\frac{dP}{dz} = -\rho g_z = -\frac{\mu P}{kT}\frac{GM}{r^2}\frac{z}{r} = -\frac{G\mu M}{kTr^3}Pz$

c. Bring all of the *P* and *z* factors to opposite sides:

$$\frac{dP}{P} = -\frac{G\mu M}{kTr^3} z dz$$
$$\int_{P_0}^{P(z)} \frac{dP'}{P} = -\frac{G\mu M}{kTr^3}\int_0^z z' dz'$$

$$\ln P(z) - \ln P_0 = \ln\left(\frac{P(z)}{P_0}\right) = -\frac{G\mu M}{kTr^3}\frac{z^2}{2} = -\frac{z^2}{H^2} \quad \text{, with } H = \left(\frac{2kTr^3}{G\mu M}\right)^{1/2} :$$

$$P(z) = P_0 e^{-z^2/H^2} \quad .$$

This functional form is called a Gaussian. You'll be seeing a lot of Gaussians, the next few years.

d $$H = \left[\frac{2kT_0 r^3}{G\mu M}\left(\frac{r_0}{r}\right)^{1/2}\right]^{1/2} = \left[\frac{2kT_0\sqrt{r_0}}{G\mu M} r^{5/2}\right]^{1/2} = \left[\frac{2kT_0 r_0^3}{G\mu M}\left(\frac{r}{r_0}\right)^{5/2}\right]^{1/2} = H_0\left(\frac{r}{r_0}\right)^{5/4} ,$$

where $H_0 = \left(2kT_0 r_0^3/G\mu M\right)^{1/2}$.

What we've done wrong is to use the temperature function which was derived for averages over the surfaces of isolated bodies. The dust disk, on the other hand, can be regarded as having an opaque surface tilted with respect to the starlight; the whole disk has to be treated in much the same way that we treated different latitudes in rapid rotators, or different distances from the subsolar point in slow rotators. If the disk is flared, then the parts more steeply flared intercept more sunlight, and will rise to a higher temperature. This will modify the dependence of temperature on orbital radius.

4. a. If you simply google "Algol eclipse" the top result is the site I always go to, Sky and Telescope Magazine's Algol Minima calculator:

http://www.skyandtelescope.com/observing/objects/variablestars/Minima_of_Algol.html

Eclipses occur every 2.87 days and the star is at its minimum brightness for about two hours. It takes another hour or two after that to recover its maximum brightness. The center (brightness minimum) of the next Algol eclipse happens tomorrow morning (or happened this morning, depending upon which recitation you're in) at 6:18 AM EST. The next three after that also happen during east-coast nighttime, like the one at 3:07 AM EST on 5 December 2011. Most will find the one at 8:46 PM on 10 December 2011 to be the most convenient to observe, clouds willing.

b. Algol is supposed to be one of the eyes of Medusa, winking at us. Medusa was a dangerous demon whose legendary frightfulness, highlighted by the nest of venomous snakes that take the place of her hair, could turn living beings to stone at a glance. Perseus snuck up on Medusa – using her reflection in his shield to avoid getting petrified himself – and beheaded her, then stuck her head in a sack and rode his winged horse Pegasus off to where Andromeda had been chained up at the beach to be sacrificed to the sea monster Cetus. Upon arrival Perseus cautioned Andromeda to shield her eyes, and then revealed the head of Medusa to Cetus, turning the monster to stone. And of course this was followed by living happily ever after.