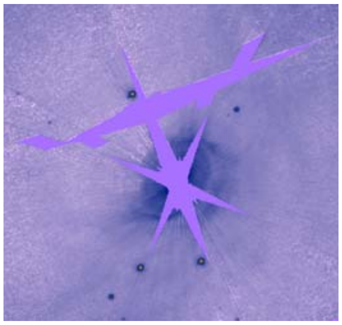
Today in Astronomy 142

- ☐ Approximations, scaling relations, and characteristic scales in astrophysics.



At right: not a spiral galaxy, but a protoplanetary disk around the Ae star HD 100546, using the coronagraphic mode of the STIS instrument on the Hubble Space Telescope (Carol Grady and coworkers, NOAO/STScI/NASA).

Approximations in the equations of astrophysics

- ☐ Astrophysical objects, be they planets, stars, nebulae or galaxies, are all very complex compared to the physical systems you have met hitherto.
- ☐ In order to simplify the relevant systems of equations that describe these objects to the point that they can be solved, astrophysicists have to employ **approximations** to the functions involved. The approximations used in introductory treatments of the subjects are often very crude, but can still be useful in illuminating the general operating features of astrophysical systems.
- ☐ Good simple approximations can often be obtained from **power-series representations** of elementary functions.

Common power-series representations and first-order approximations

Suppose $x \ll 1$. Then x^2, x^3, *etc.* are even smaller. A **first-order** approximation is one in which we ignore power-series terms of higher power than x^1. Examples:

$$\sin x = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots \cong x$$

$$\cos x = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!} = 1 - \frac{x^2}{2} + \frac{x^4}{120} - \ldots \cong 1$$

$$\tan x = \sum_{i=0}^{\infty} \frac{2^{2i+2}\left(2^{2i+2}-1\right)B_i x^{2i+1}}{(2i+2)!} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots \cong x$$

$$\arctan x = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{2i+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \cong x$$

Small-angle approximation

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2} + \ldots \cong 1 + x$$

$$\ln(1+x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^{i+1}}{i+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \cong x$$

$$(1+x)^n = \sum_{i=0}^{\infty} \frac{n!}{i!(n-i)!} x^i = 1 + nx + \frac{n(n-1)}{2} x + \ldots \cong 1 + nx$$

Example first-order approximations

Find approximations to first order in x for:

$$\sqrt{e^x \cos x}$$

$$\frac{1}{e^x - 1}$$

$$\frac{4^n \tan x}{(2+x)^n (2-x)^n}$$

$$\frac{e^{ix} - e^{-ix}}{2i}$$

Example first-order approximations

Find approximations to first order in x for:

Answers:

$$\sqrt{e^x \cos x} \qquad \cong \sqrt{(1+x)1} \cong 1 + \frac{x}{2}$$

$$\frac{1}{e^x - 1} \qquad \cong \frac{1}{1+x-1} = \frac{1}{x}$$

$$\frac{4^n \tan x}{(2+x)^n (2-x)^n} \qquad = \frac{4^n \tan x}{\left(4 - x^2\right)^n} \cong \frac{4^n \tan x}{4^n} \cong x$$

$$\frac{e^{ix} - e^{-ix}}{2i} \qquad \cong \frac{1 + ix - 1 - (-ix)}{2i} = x$$

Scaling relations

Sometimes the difference between results under different approximations or assumptions takes the form of common function of some key physical parameters times different factors that are independent of these parameters. In this case the cruder approximation gives us a useful **scaling relation.**

- ☐ **Example**
  Mass density at center of *uniform* sphere with mass M and radius R:

$$\rho_0 = \frac{M}{V} = \frac{3}{4\pi} \frac{M}{R^3}$$

Scaling relations (continued)

Mass density at center of sphere with mass M and density that varies according to $\rho(r) = \rho_0 e^{-r/R}$:

$$M = \int \rho dV = \int_0^\infty \rho(r) 4\pi r^2 dr = 4\pi\rho_0 \int_0^\infty r^2 e^{-r/R} dr$$

$$= 4\pi\rho_0 R^3 \int_0^\infty u^2 e^{-u} du \quad \text{Integrate by parts twice...}$$

$$= 8\pi\rho_0 R^3 \quad ; \text{ that is,}$$

$$\rho_0 = \frac{1}{8\pi}\frac{M}{R^3} \quad .$$

Scaling relations

It looks as if the mass density at the center of the sphere would always have the form

$$\rho_0 = \begin{bmatrix} \text{factor independent of} \\ \text{mass and radius} \end{bmatrix} \times \frac{M}{R^3}$$

no matter what the details of the density. Common astrophysical nomenclature:

$$\rho_0 \propto \frac{M}{R^3}$$

("central density is proportional to, or **scales with**, M/R^3").

Scaling relations (concluded)

Why is this scaling relation useful?

- ❑ Consider that it means that, even if you don't know how the density actually varies with radius in this sphere, you would know that the central density probably changes by a factor of 8 if the radius of the sphere changes by a factor of 2, and that the central density probably changes by a factor of 2 if the mass changes by a factor of 2.

Characteristic scales

- ❑ Note that the sphere doesn't have a sharp edge in the exponential-density case. Thus R is not "the" radius of the sphere in this case; it is a radius **characteristic** of the material in the sphere.