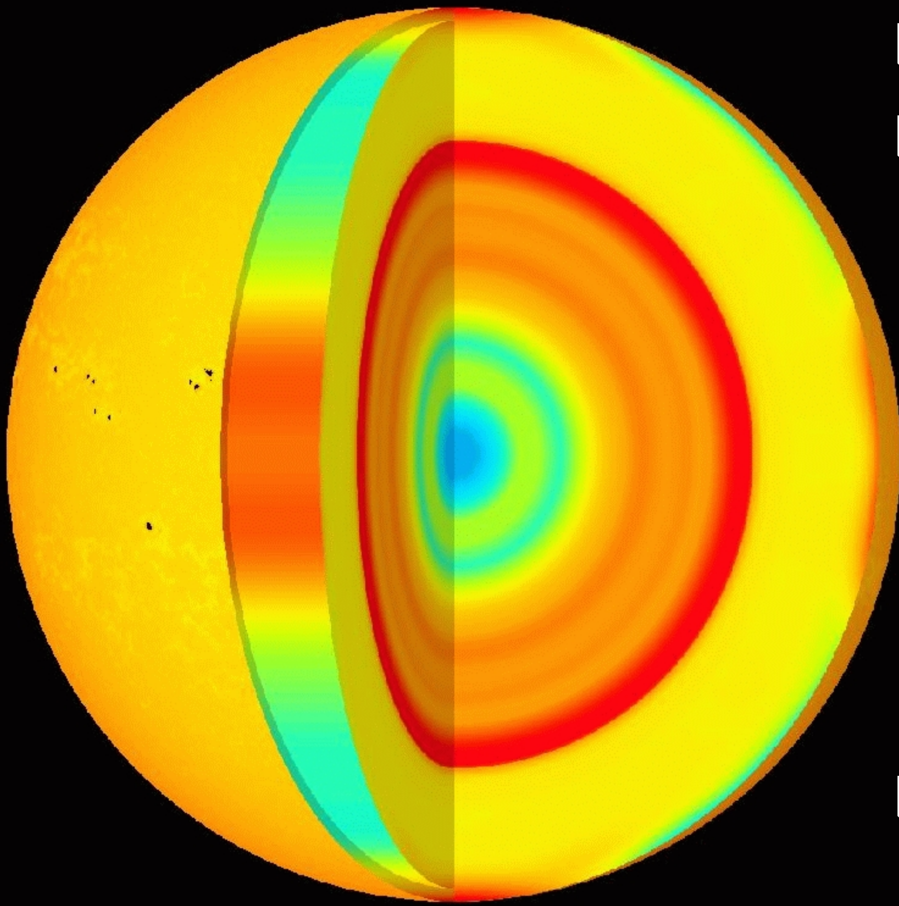
Today in Astronomy 142: stellar structure



- ❑ Principles of stellar structure
- ❑ Hydrostatic equilibrium gives scaling relations for
  - average pressure, density and temperature
  - pressure, density and temperature in the center (where the luminosity is produced)
- ❑ Diffusion of light from center to surface.

Rotational structure of the Sun: the different colors indicate flows slower (blue) and faster (red) than average. (SOHO/NASA).

Principles of stellar structure

- Vogt-Russell "theorem:" **the mass and composition of a star uniquely determine its radius, luminosity, internal structure, and subsequent evolution**.
- Stars are **spherical**, to good approximation.
- Stable stars are in **hydrostatic equilibrium**: the weight of each infinitesimal piece of the star's interior is balanced by the force from the pressure difference across the piece.
  - Most of the time the pressure is gas pressure, and is described well in terms of density and temperature by the **ideal gas law**, $PV = NkT$.
  - However, in very hot stars or giant stars, the pressure exerted by light -- radiation pressure -- can dominate.

Principles of stellar structure (continued)

- Energy is transported from the inside to the outside, most of the time in the form of **light**.
  - The interiors of stars are **opaque**. Photons are absorbed and reemitted many many times on their way from the center to the surface: a random-walk process called **diffusion**.
  - The opacity depends upon the density, temperature and chemical composition.
  - Most stars have regions in their interiors in which the radial variations of temperature and pressure are such that hot bubbles of gas can “boil” up toward the surface. This process, called **convection**, is a very efficient energy-transport mechanism and can be more important than light diffusion in many cases.

The equations of stellar structure

Hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{GM_r \rho}{r^2}$$

Mass conservation:

$$\frac{dM_r}{dr} = 4\pi r^2 \rho$$

Energy generation:

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \varepsilon$$

Energy transport:

$$\frac{dT}{dr} = -\frac{3}{16\sigma}\frac{\bar{\kappa}\rho}{r^2}\frac{L_r}{4\pi r^2}$$

Adiabatic temperature gradient:

$$\left(\frac{dT}{dr}\right)_{\text{rad}} = -\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2}$$

Convection occurs if:

$$\frac{T}{P}\frac{dP}{dT} < \frac{\gamma}{\gamma - 1}$$

Equations of state

Pressure:

$P = P(\rho, T, \text{composition})$ in general

$= \dfrac{\rho kT}{\mu m_H} + \dfrac{4\sigma T^4}{3c}$ throughout most normal stars

Opacity:

$\bar{\kappa} = \bar{\kappa}(\rho, T, \text{composition})$ in general

Energy generation:

$\varepsilon = \varepsilon(\rho, T,$ composition) in general

Boundary conditions:

$$\left.\begin{matrix} M_r \to 0 \\ L_r \to 0 \end{matrix}\right\} \text{ as } r \to 0$$

$$\left.\begin{matrix} T \to 0 \\ P \to 0 \\ \rho \to 0 \end{matrix}\right\} \text{ as } r \to R_{\text{star}}$$

M_r, L_r: mass or luminosity contained within radius r.

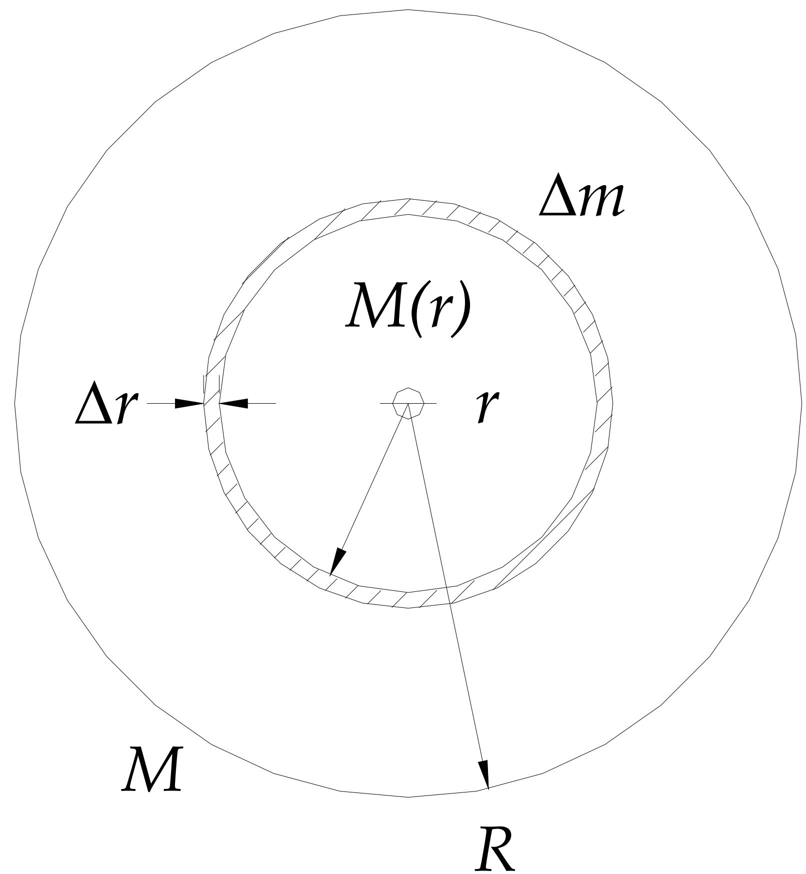
The equations of stellar structure (continued)

- **We show these here for illustration only**; we will not attempt to solve this network of differential equations. (Take Astronomy 241 to learn some "simple" solutions, Astronomy 453 to learn how to do it properly.)
- In fact, there are no general, analytical solutions to the equations; usually stellar-interior models are generated by computer solution.
- Instead of solving these equations, we shall employ some crude approximations to develop several **scaling relationships** that will also be crude, but will correctly illuminate the physical processes important in stars.
- But the equation of **hydrostatic equilibrium** is useful by itself, so we will take the time here to derive it.

Hydrostatic equilibrium

Consider a spherical shell of radius r and thickness $\Delta r \ll r$ within a star with mass density (mass per unit volume) $\rho(r)$. Its weight is

$$\begin{aligned}\Delta F &= -\frac{GM(r)\Delta m}{r^2} \\ &= -\frac{GM(r)}{r^2}4\pi r^2 \Delta r\, \rho(r) \\ &= -4\pi GM(r)\,\rho(r)\,\Delta r \quad .\end{aligned}$$



(minus sign: force points inward)

Hydrostatic equilibrium (continued)

If the star is in hydrostatic equilibrium the weight is balanced by the pressure difference across the shell:

$$-\Delta F = P(r)4\pi r^2 - P(r+\Delta r)4\pi(r+\Delta r)^2 \qquad (\Delta r \ll r)$$

$$\cong P(r)4\pi r^2 - \left[P(r) + \frac{dP}{dr}\Delta r\right]4\pi r^2 \qquad \text{(to first order in pressure)}$$

$$= -\frac{dP}{dr}4\pi r^2 \Delta r \quad ,$$

so $$\frac{dP}{dr} = \frac{\Delta F}{4\pi r^2 \Delta r} = -\frac{4\pi G M(r)\rho(r)\Delta r}{4\pi r^2 \Delta r} = \boxed{-\frac{GM(r)\rho(r)}{r^2}} \quad .$$

This differential equation, one of the equations of stellar structure, can be solved easily under certain assumptions.

The Sun’s interior, on the average

Since we know its distance (from radar), mass (from Earth’s orbital period) and radius (from angular size and known distance):

$$r = 1\ \mathrm{AU} = 1.4960 \times 10^{13}\ \mathrm{cm}$$

$$M_\odot = 1.98843 \times 10^{33}\ \mathrm{gm}$$

$$R_\odot = 6.9599 \times 10^{10}\ \mathrm{cm}$$

we know the average mass density (mass per unit volume):

$$\bar{\rho}_\odot = \frac{M_\odot}{V_\odot} = \frac{3M_\odot}{4\pi R_\odot^3} = 1.41\ \mathrm{g\ cm^{-3}}$$

= only 26% of Earth's.

The Sun's interior, on the average (continued)

For the average pressure, we use the equation of hydrostatic equilibrium, and assume (*very* crudely) that the gas pressure varies linearly from some central value to zero at the surface:

$$\frac{dP}{dr} \approx \frac{\Delta P}{\Delta r} = \frac{P_{\text{surface}} - P_{\text{center}}}{r_{\text{surface}} - r_{\text{center}}} = -\frac{P_{\text{center}}}{R_\odot}$$

$$= -\frac{GM\rho}{r^2} = -\frac{GM_\odot \bar{\rho}_\odot}{R_\odot^2}$$

Then $$\bar{P}_\odot = \frac{P_{\text{center}}}{2} = \frac{GM_\odot \bar{\rho}_\odot}{2R_\odot}$$

$$= \frac{\left(6.67\times10^{-8}\right)\left(1.99\times10^{33}\right)1.41}{2\times6.96\times10^{10}} \text{ dyne cm}^{-2}$$

$$= 1.34\times10^{15} \text{ dyne cm}^{-2} = 1.3\times10^{9} \text{ atmospheres}$$

The Sun's interior, on the average (continued)

The Sun is an ideal gas:

$$PV = NkT$$

$$P = nkT = \frac{\rho kT}{\overline{m}}$$

(N= number of gas particles, n= number density: number of particles per unit volume, $\overline{m}$= avg. particle mass)

Average particle mass is about the mass of the proton (1.67×10^{-24} gm), so

$$\overline{T}_\odot = \frac{\overline{m}\overline{P}_\odot}{\overline{\rho}_\odot k} = \frac{\left(1.67 \times 10^{-24}\right)\left(1.34 \times 10^{15}\right)}{(1.41)\left(1.38 \times 10^{-16}\right)} \mathrm{K}$$

$$= 11.5 \times 10^{6} \mathrm{~K} \quad .$$

Now for some less-crude estimates….

Central pressure in a star

The star's center, being its densest and hottest spot, will turn out to be the site of virtually all of the star's energy generation, so we will make somewhat more careful estimates of the conditions there.

From hydrostatic equilibrium equation again:

$$P(R) - P(r) = \int_r^R \frac{dP}{dr'} dr' = -\int_r^R \frac{GM(r')\rho(r')}{r'^2} dr' \quad .$$

But the pressure at the surface, $P(R)$, better be zero because the star's surface doesn't move and there's nothing outside the surface to push back, so

$$P(r) = \int_r^R \frac{GM(r')\rho(r')}{r'^2} dr' \quad .$$

Central pressure in a star (continued)

We can't do the integral unless we know the density as a function of position, so instead we make a crude approximation:

$$\rho \approx \frac{M}{V} \approx \frac{M}{R^3}$$

(ignoring dimensionless factors like $4\pi/3$, because we're just trying to get the order of magnitude right),

and the integral becomes

$$P(r) \approx \frac{GM}{R^3}\int_r^R \frac{M(r')}{r'^2}dr' \approx \frac{GM}{R^3}\int_r^R \frac{1}{r'^2}\left(M\frac{r'^3}{R^3}\right)dr'$$

$$\approx \frac{GM^2}{R^6}\int_r^R r'dr'$$

Central pressure in a star (continued)

Central pressure (r = 0):

$$P_C \approx \frac{GM^2}{R^6}\int_0^R r' dr' = \frac{GM^2}{R^6}\frac{R^2}{2}$$

$$\approx \boxed{\frac{GM^2}{R^4}} \quad \text{(still ignoring dimensionless factors)}$$

Lo and behold, a complete calculation for stars of moderate to low mass (Astronomy 453 style) yields

$$P_C = 19\frac{GM^2}{R^4}$$

so we have derived a pretty good scaling relation for P_C.

Central pressure in a star (continued)

For the Sun:

$$M_\odot = 1.99 \times 10^{33} \text{ g} \qquad R_\odot = 6.96 \times 10^{10} \text{ cm}$$

$$P_C \cong 19 \frac{GM_\odot^2}{R_\odot^4} = 2.1 \times 10^{17} \text{ dyne cm}^{-2}$$

$$> 10^{11} \text{ atmospheres}$$

So, for other main sequence stars, to adequate approximation,

$$P_C \cong 19 \frac{GM^2}{R^4} = 2.1 \times 10^{17} \left(\frac{M}{M_\odot} \right)^2 \left(\frac{R}{R_\odot} \right)^4 \text{ dyne cm}^{-2} \quad .$$

Central pressure in a star (continued)

That is,

$$P_C \cong 19\frac{GM^2}{R^4} = 19\frac{GM^2}{R^4}\left(\frac{M_\odot}{M_\odot}\right)^2\left(\frac{R_\odot}{R_\odot}\right)^4$$

$$= 19\frac{GM_\odot^2}{R_\odot^4}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R}{R_\odot}\right)^4$$

$$= 19\frac{\left(6.67\times10^{-8}\ \mathrm{cm\ gm^{-1}\ sec^{-2}}\right)\left(1.99\times10^{33}\ \mathrm{gm}\right)^2}{\left(6.96\times10^{10}\ \mathrm{cm}\right)^4}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R}{R_\odot}\right)^4$$

$$= 2.1\times10^{17}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R}{R_\odot}\right)^4\ \mathrm{dyne\ cm^{-2}} \quad .$$

(You'll need to get used to doing this.)

Central density and temperature of the Sun

Central pressure is a little more than a factor of 100 larger than average pressure. Guess: central density 100 times higher than average? (That's equivalent to guessing that the internal temperature does vary much with radius.)

- For the Sun, that's not bad; the central density turns out to be 110 times the average density, $\rho_C = 150 \text{ gm cm}^{-3}$. Thus, since $\rho_C \propto M/R^3$,

$$\rho_C = 25\frac{M}{R^3} = 150\left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right)^3 \text{ gm cm}^{-3}$$

- As we will see in a couple of weeks, the average gas-particle mass in the center of the Sun, considering its composition and the fact that the center is completely ionized, is $\overline{m}_C = 1.5\times 10^{-24} \text{ gm}$.

Central density and temperature of the Sun (continued)

- But the material is still an ideal gas, so

$$T_C = \frac{P_C V}{N_C k} = \frac{P_C}{n_C k} = \frac{P_C \bar{m}_C}{\rho_C k} = 15.7 \times 10^6 \text{ K}.$$

  Compare to $\bar{T}_\odot$: T doesn't vary very much with radius.
- We can make a scaling relation out of this as well, to use in extrapolating to stars similar to the Sun but having different masses, sizes and composition:

$$T_C = \frac{P_C \bar{m}_C}{\rho_C k} \propto \frac{GM^2}{R^4} \frac{R^3}{M} \bar{m}_C = 15.7 \times 10^6 \text{ K for the Sun;}$$

$$T_C = 15.7 \times 10^6 \left(\frac{M}{M_\odot} \right) \left(\frac{R_\odot}{R} \right) \left(\frac{\bar{m}}{\bar{m}_\odot} \right) \text{ K} \quad .$$

Opacity and luminosity in stars

At the high densities and temperatures found on average in stellar interiors, matter is opaque. The **mean free path**, or average distance a photon can travel before being absorbed, is about $\ell = 0.5$ cm for the Sun's average density and temperature (given above).

Photons produced in the center have to random-walk their way out. **How many steps, or mean free paths, does it take for a photon to random-walk from center to surface?** (See PU problem 5.11.)

- Suppose photon starts off at the center of the star, and has an equal chance to go right or left after each absorption and re-emission. Average value of position after N steps is

$$\langle x_N \rangle = (x_1 + x_2 + \cdots + x_N) / N = 0$$

Opacity and luminosity in stars (continued)

- However, the average value of the square of the position is not zero. Consider step N+1, assuming the chances of going left or right are equal:

$$\begin{aligned}\left\langle x_{N+1}^2 \right\rangle &= \frac{1}{2}\left\langle \left(x_N - \ell\right)^2 \right\rangle + \frac{1}{2}\left\langle \left(x_N + \ell\right)^2 \right\rangle \\ &= \frac{1}{2}\left\langle x_N^2 - \cancel{2x_N\ell} + \ell^2 \right\rangle + \frac{1}{2}\left\langle x_N^2 + \cancel{2x_N\ell} + \ell^2 \right\rangle \\ &= \left\langle x_N^2 \right\rangle + \ell^2 \quad .\end{aligned}$$

- But if this is true for all N, then we can find $\left\langle x_N^2 \right\rangle$ by starting at zero and adding ℓ^2, N times (i.e. using induction):

$$\left\langle x_N^2 \right\rangle = N\ell^2 \quad .$$

Opacity and luminosity in stars (continued)

- Thus to random-walk a distance $\sqrt{\left\langle x_N^2 \right\rangle} = L$, the photon needs to take on the average

$$N = \frac{L^2}{\ell^2} \quad \text{steps.}$$

- So far, we have discussed only one dimension of a three-dimensional random walk. Three times as many steps need to be taken in this case, so to travel a distance *R*, the photon on the average needs to take

$$N = \frac{3R^2}{\ell^2} \quad \text{steps.}$$

Opacity and luminosity in stars (continued)

- For the Sun, and for a constant mean free path of 0.5 cm,

$$N = \frac{3\left(6.96\times 10^{10}\right)^2}{(0.5)^2} = 5.81\times 10^{22} \text{ steps.}$$

  (Very opaque indeed.)

- Each step takes a time $\Delta t = \ell / c$, so the average time it takes for a photon to diffuse from the center of the Sun to the surface is

$$t = N\Delta t = \frac{3R_\odot^2}{\ell c} = 9.7\times 10^{11} \text{ s} = 3.1\times 10^4 \text{ years.}$$

  Note that the same trip only takes $R_\odot / c = 2.3$ s for a photon travelling in a straight line.