Today in Astronomy 142: models of a matter-dominated universe

Origin of Birkhoff's Rule, and "simple" results for matter-dominated universes:

- ❑ Is the Universe gravitationally bound?
- ❑ How old is the Universe?

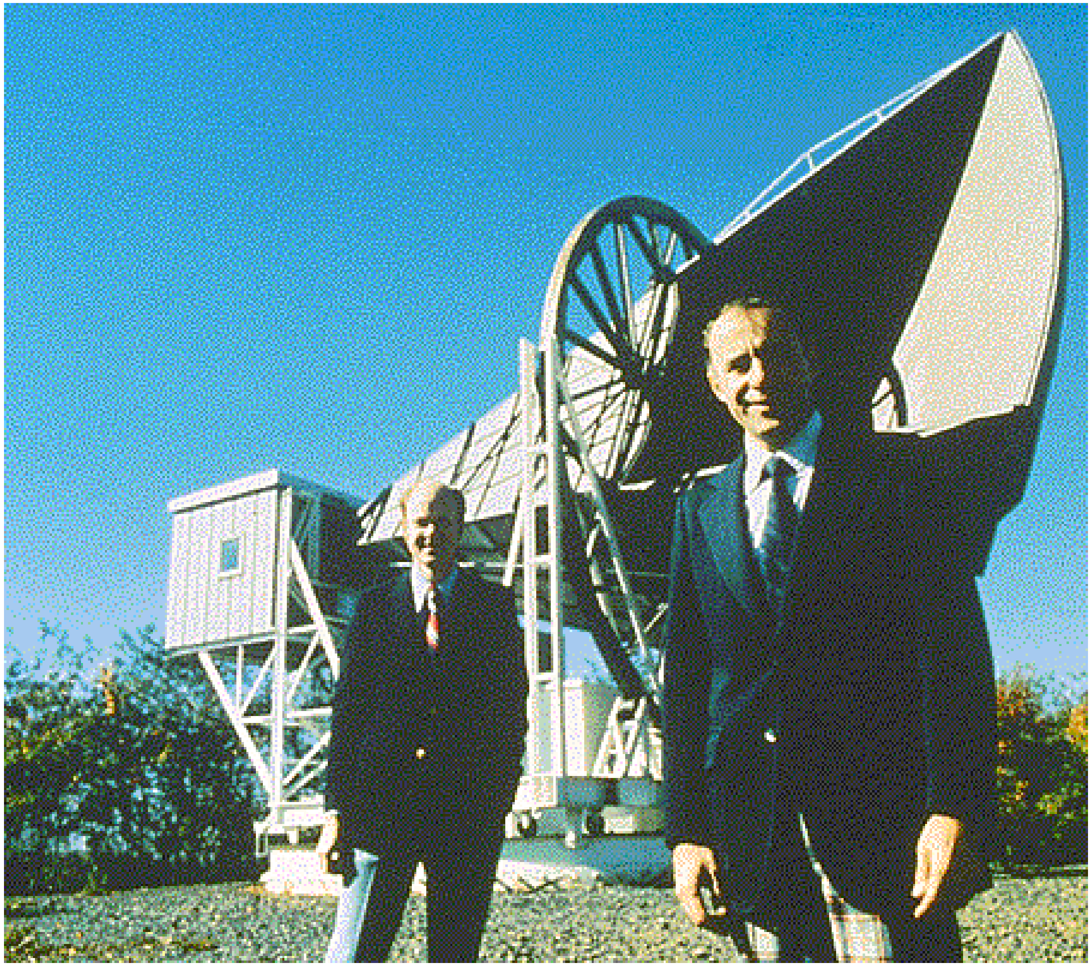


Image: Bob Wilson (left) and Arno Penzias (right) with the horn antenna they used to discover the cosmic microwave background.

Warning!

This lecture may seem to some of you to include the most difficult material of the semester. So it's worth noting that

- it's not *really* that hard, it's just a lot of writing.
- it's all downhill from here, for the rest of the semester.

Origin of Birkhoff's Rule

To illuminate further the Robertson-Walker interval and the terms it contains, we will now show how Birkhoff's rule falls out of one component of the Einstein field equations (!).

Here it is:

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G}{3}\rho_m - \Lambda = -K\frac{c^2}{R^2}$$

Mass density. (ρ_m)

Cosmological constant (Λ)**.**
Not originally a part of general relativity. We'll take it to be zero now, but it will come up again next week. Thus we deal now with a **matter-dominated Universe**.

Origin of Birkhoff's Rule (continued)

Multiply through by $R^2 r_*^2$, divide by 2:

$$\frac{1}{2}\dot{R}^2 r_*^2 - \frac{4\pi G R^2 r_*^2}{3}\rho_m = -K\frac{c^2 r_*^2}{2}$$

But $Rr_* = r$ and $\dot{R} r_* = \dot{r} = \frac{dr}{dt}$, so

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{4\pi G r^2}{3}\rho_m = -K\frac{c^2 r_*^2}{2}$$

Take r to be a large distance from us; the mass M contained in a sphere of radius r is

$$M = \frac{4\pi r^3}{3}\rho_m$$

Origin of Birkhoff's Rule (continued)

Thus $$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} = -K\frac{c^2 r_*^2}{2} \quad (= \text{ constant})$$

Now consider a test particle, like a galaxy, with mass m located on the sphere. Multiply the equation through by m and note that the speed of the particle is $v = dr/dt$:

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = -K\frac{mc^2 r_*^2}{2}$$

$$E =$$

Evidently the constant on the right is the test particle's total energy, and the dynamics of this particle is determined only by the mass within the sphere (thus demonstrating Birkhoff's rule).

Age and fate of the Universe

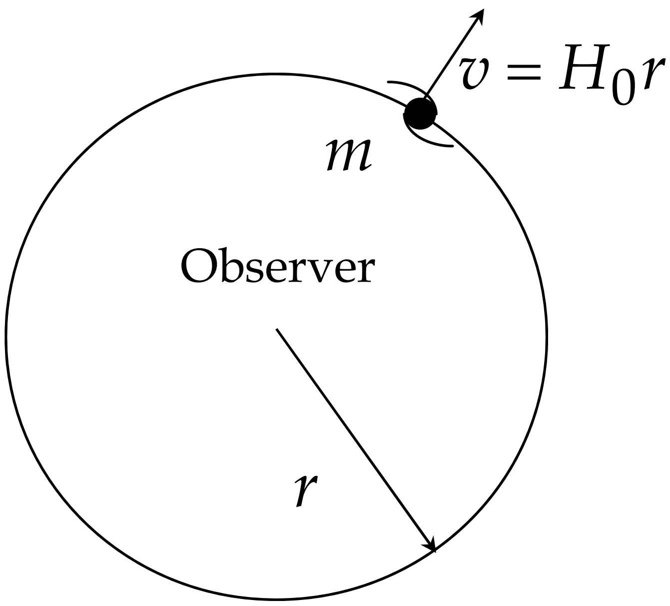
We will now do a couple of calculations, using Birkhoff's Rule and the R-W interval, to get answers to these questions of basic importance in our description of the Universe:

- Will the present expansion of the Universe continue forever (solution 4), or will it stop (3), or might it even reverse and collapse again (2)?
- How old is the Universe?

These calculations aren't very hard but some require a lot of writing, and get quite complicated-looking. We'll skip over the tedious complex parts, just illustrating how they would go by doing the easy parts in detail.

Is the Universe gravitationally bound?

Consider again that sphere and galaxy:



$$M(r) = \rho_{m0} \frac{4\pi r^3}{3}$$

Energy of galaxy on sphere, according to Birkhoff's Rule:

$$E = \frac{1}{2} m v^2 - \frac{GmM(r)}{r}$$

$E = 0$ divides bound and unbound universes: $E > 0$ means that the Universe will expand forever, $E < 0$ means that it will eventually collapse.

Is the Universe gravitationally bound? (continued)

For the critical (or "marginal") universe (E = 0),

$$\frac{Gm}{r}\frac{4\pi r^3}{3}\rho_{m0} = \frac{1}{2}mv^2 = \frac{1}{2}mH_0^2 r^2$$

$$\rho_{m0} = \frac{3H_0^2}{8\pi G} = 7.9\times 10^{-30}\ \mathrm{gm\ cm^{-3}} \qquad \text{Critical density}$$

$$\Omega = \frac{\rho}{\rho_{m0}} \qquad \text{Normalized density ("omega")}$$

If $\Omega < 1$, the universe is not gravitationally bound (**"open universe"**); if $\Omega > 1$, it is (**"closed universe"**). Which Universe do we live in? We must consult observations…

Is the Universe gravitationally bound? (continued)

Observational bounds on Ω, made from galaxy redshift surveys over the past 30-35 years, using galaxies as test particles, and thus sensitive to luminous and dark matter:

$$\Omega = 0.2 \pm 0.1$$

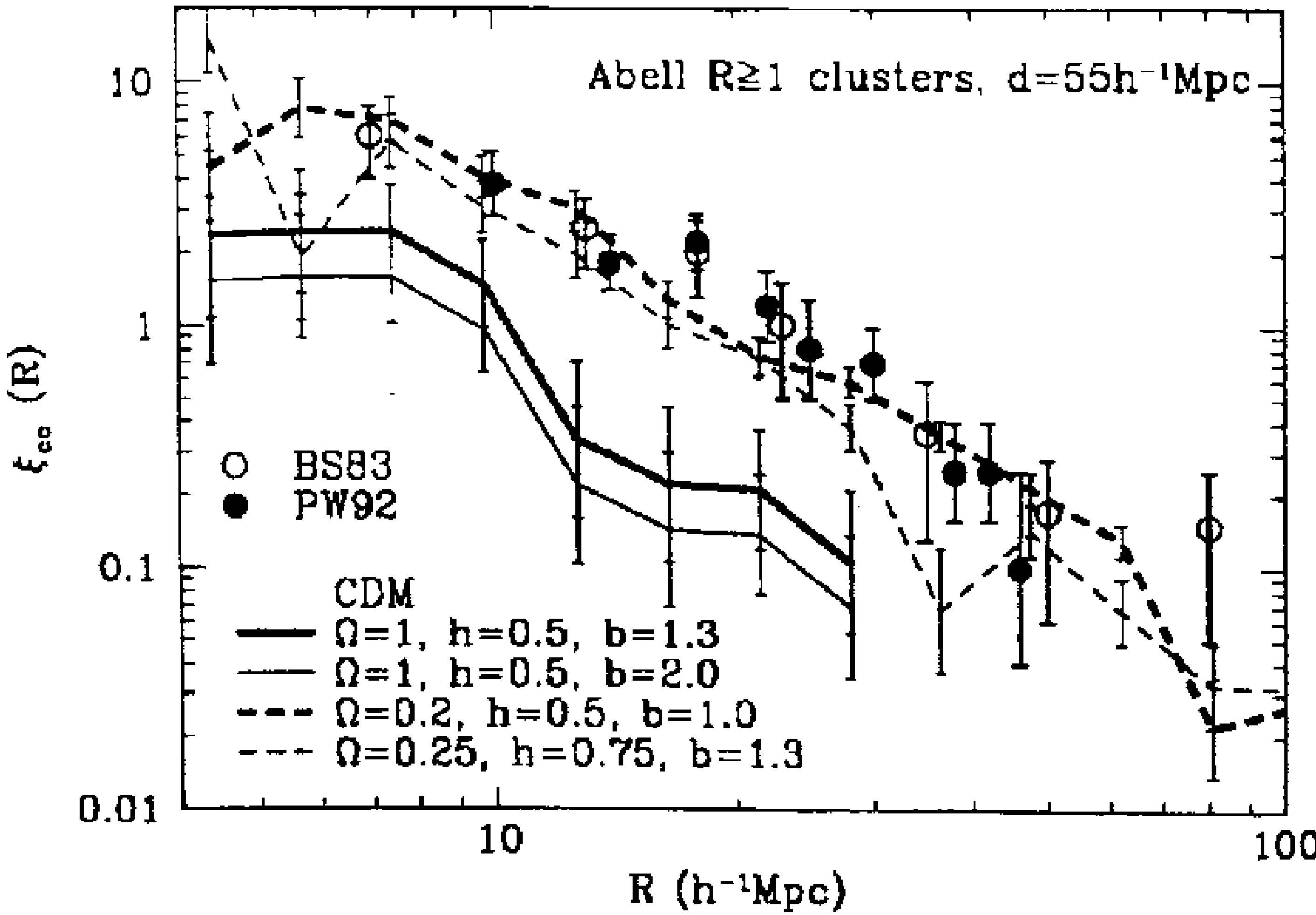
(e.g. Dressler, Faber, N. Bahcall, Huchra, Geller, …).

On the other hand, strong theoretical arguments, based on "inflationary" models of the early Universe and presented over the same period, indicate that we should get

$$\Omega \equiv 1$$

if the Universe is matter-dominated (e.g. Gott, Guth, Schramm, Kolb,…).

Is the Universe gravitationally bound? (continued)

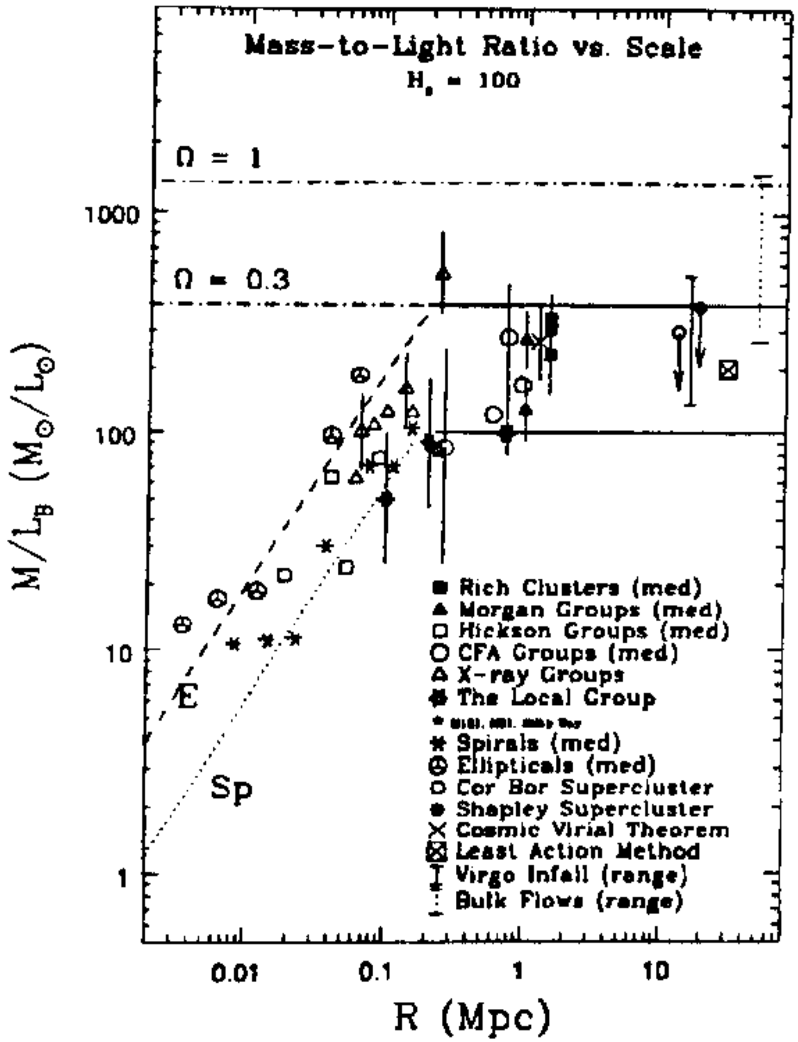


Galaxy cluster correlation function, compared with expectations for different Ω and

$$h = \frac{H_0}{100 \text{ km s}^{-1} \text{ Mpc}^{-1}}$$

(N. Bahcall 1997)

Is the Universe gravitationally bound? (continued)



Summary of dark matter and measurements of the Universe’s mass density (N. Bahcall 1997)

How old is the Universe?

The density of the Universe declines with time as the Univese expands. But energy is conserved; thus the expansion rate (Hubble "constant") must vary with time unless the Universe is empty (though it's constant throughout the Universe at any given time).

- Conversely the density is higher at earlier times. At what time was the density infinite? (The time elapsed since then is taken to be the age of the Universe.)
- Birkhoff's rule and the energy of the test galaxy a distance r away give us a differential equation we can solve for the age of the Universe.

How old is the Universe? (continued)

$$E = \frac{1}{2}mv^2 - \frac{GmM}{r} = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 - \frac{GmM}{r}$$

If $E = 0$ (Ω=1), then $\frac{dr}{dt} = \sqrt{\frac{2GM}{r}}$.

Positive root chosen for expanding solution.

Separate and integrate, using $r = 0$ at $t = 0$:

$$\int_0^{r_0} \sqrt{r}\,dr = \sqrt{2GM}\int_0^{t_0} dt \quad \Rightarrow \quad \frac{2}{3}r_0^{3/2} = t_0\sqrt{2GM}$$

$$\Rightarrow \quad t_0 = \frac{2}{3}\sqrt{\frac{r_0^3}{2GM}}$$

How old is the Universe? (continued)

But for the present day (zero subscripts) we can write

$$E = \frac{1}{2}mv_0^2 - \frac{GmM}{r_0} = \frac{1}{2}mH_0^2 r_0^2 - \frac{GmM}{r_0} = 0$$

since energy is conserved. Solving this for the Hubble constant,

$$H_0 = \sqrt{\frac{2GM}{r_0^3}}$$

we see that if the Universe is critical then its age is

$$t_0 = \frac{2}{3H_0} = \frac{2}{3\left(64 \text{ km sec}^{-1} \text{ Mpc}^{-1}\right)}\left(\frac{3.09\times10^{19} \text{ km}}{1 \text{ Mpc}}\right)\left(\frac{\text{year}}{3.16\times10^{7} \text{ sec}}\right)$$

$$= 1.0\times10^{10} \text{ years}$$

How old is the Universe? (continued)

Now for open and closed universes. Cancel the m:

$$E = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{4\pi G\rho_m}{3}r^2 = \frac{1}{2}H_0^2 r_0^2 - \frac{4\pi G\rho_{m0}}{3}r_0^2$$

Define

$$D_* = r/r_0 = R(t)/R(t_0) \qquad \Rightarrow \qquad \frac{dr}{dt} = \frac{dr}{dD_*}\frac{d\tau_*}{dt}\frac{dD_*}{d\tau_*} = r_0 H_0 \frac{dD_*}{d\tau_*}$$

$$\tau_* = H_0 t$$

and divide through by $\frac{1}{2}H_0^2 r_0^2$:

$$\left(\frac{dD_*}{d\tau_*}\right)^2 - \frac{8\pi G\rho_m}{3H_0^2}\frac{r^2}{r_0^2} = 1 - \frac{8\pi G\rho_{m0}}{3H_0^2}$$

How old is the Universe? (continued)

Define the present, normalized mass density of the Universe:

$$\Omega_0 = \frac{8\pi G \rho_{m0}}{3H_0^2} \quad ,$$

note that $\frac{4\pi}{3}\rho_m r^3 = \frac{4\pi}{3}\rho_m r_0^3$, and we get

$$\left(\frac{dD_*}{d\tau_*}\right)^2 - \frac{\Omega_0}{D_*} = 1 - \Omega_0$$

More substitutions:

$$\xi = \frac{|1-\Omega_0|}{\Omega_0} D_* \qquad \tau = \frac{|1-\Omega_0|^{3/2}}{\Omega_0}\tau_*$$

$$\frac{dD_*}{d\tau_*} = \frac{dD_*}{d\xi}\frac{d\tau}{d\tau_*}\frac{d\xi}{d\tau} = \frac{\Omega_0}{|1-\Omega_0|}\frac{|1-\Omega_0|^{3/2}}{\Omega_0}\frac{d\xi}{d\tau} = \sqrt{|1-\Omega_0|}\,\frac{d\xi}{d\tau}$$

How old is the Universe? (continued)

Thus
$$\left|1-\Omega_0\right|\left(\frac{d\xi}{d\tau}\right)^2-\Omega_0\,\frac{\left|1-\Omega_0\right|}{\Omega_0}\frac{1}{\xi}=1-\Omega_0$$

$$\left(\frac{d\xi}{d\tau}\right)^2-\frac{1}{\xi}=\frac{1-\Omega_0}{\left|1-\Omega_0\right|}=\pm1\;(=-K)$$

+: open universe
-: closed universe

Now separate and integrate:

$$\frac{d\xi}{d\tau}=\sqrt{\pm1+\frac{1}{\xi}}=\sqrt{\frac{1\pm\xi}{\xi}}$$

$$\tau=\int_0^{\tau}d\tau'=\int_0^{\xi}d\xi'\sqrt{\frac{\xi'}{1\pm\xi'}}$$

To proceed, different substitutions are required for the different signs.

How old is the Universe? (continued)

Minus sign (closed universe): let

$$\xi = \sin^2\left(\frac{\eta}{2}\right) \qquad d\xi = 2\sin\left(\frac{\eta}{2}\right)\cos\left(\frac{\eta}{2}\right)\frac{1}{2}d\eta$$

$$\sqrt{\frac{\xi}{1-\xi}} = \frac{\sin(\eta/2)}{\sqrt{1-\sin^2(\eta/2)}} = \frac{\sin(\eta/2)}{\cos(\eta/2)}$$

so

$$\tau = \int_0^\eta d\eta' \sin\left(\frac{\eta'}{2}\right)\cos\left(\frac{\eta'}{2}\right)\frac{\sin(\eta'/2)}{\cos(\eta'/2)} = \int_0^\eta d\eta' \sin^2\left(\frac{\eta'}{2}\right)$$

$$= \int_0^\eta d\eta' \frac{1-\cos\eta'}{2} = \frac{1}{2}(\eta - \sin\eta) \quad .$$

This, in fact, we have done before, on 3 March 2009, in connection with star formation.

How old is the Universe? (continued)

Plus sign (open universe): let

$$\xi = \sinh^2\left(\frac{\eta}{2}\right) \qquad d\xi = 2\sinh\left(\frac{\eta}{2}\right)\cosh\left(\frac{\eta}{2}\right)\frac{1}{2}d\eta$$

$$\sqrt{\frac{\xi}{1+\xi}} = \frac{\sinh(\eta/2)}{\sqrt{1+\sinh^2(\eta/2)}} = \frac{\sinh(\eta/2)}{\cosh(\eta/2)}$$

so

$$\tau = \int_0^\eta d\eta' \sinh\left(\frac{\eta'}{2}\right)\cosh\left(\frac{\eta'}{2}\right)\frac{\sinh(\eta'/2)}{\cosh(\eta'/2)} = \int_0^\eta d\eta' \sinh^2\left(\frac{\eta'}{2}\right)$$

$$= \int_0^\eta d\eta' \frac{\cosh\eta' - 1}{2} = \frac{1}{2}(\sinh\eta - \eta) \quad .$$

How old is the Universe? (continued)

Thus a closed universe has

$$\xi = \sin^2\left(\frac{\eta}{2}\right) = \frac{1}{2}(1-\cos\eta) = \frac{|1-\Omega_0|}{\Omega_0}D_* = \frac{|1-\Omega_0|}{\Omega_0}\frac{r}{r_0}$$

$$\tau = \frac{1}{2}(\eta - \sin\eta) = \frac{|1-\Omega_0|^{3/2}}{\Omega_0}\tau_* = \frac{|1-\Omega_0|^{3/2}}{\Omega_0}H_0 t \quad ,$$

or

$$D_* = \frac{r}{r_0} = \frac{\Omega_0}{2|1-\Omega_0|}(1-\cos\eta)$$ Maximum for $\cos\eta = -1$

$$t = \frac{\Omega_0}{2H_0|1-\Omega_0|^{3/2}}(\eta - \sin\eta)$$

Current age is $t_0 = t(\eta)$ for which $r(\eta) = r_0$.

How old is the Universe? (continued)

…and an open universe has

$$\xi = \frac{1}{2}(\cosh\eta - 1) = \frac{|1-\Omega_0|}{\Omega_0} D_* = \frac{|1-\Omega_0|}{\Omega_0}\frac{r}{r_0}$$

$$\tau = \frac{1}{2}(\sinh\eta - \eta) = \frac{|1-\Omega_0|^{3/2}}{\Omega_0}\tau_* = \frac{|1-\Omega_0|^{3/2}}{\Omega_0} H_0 t \quad ,$$

or

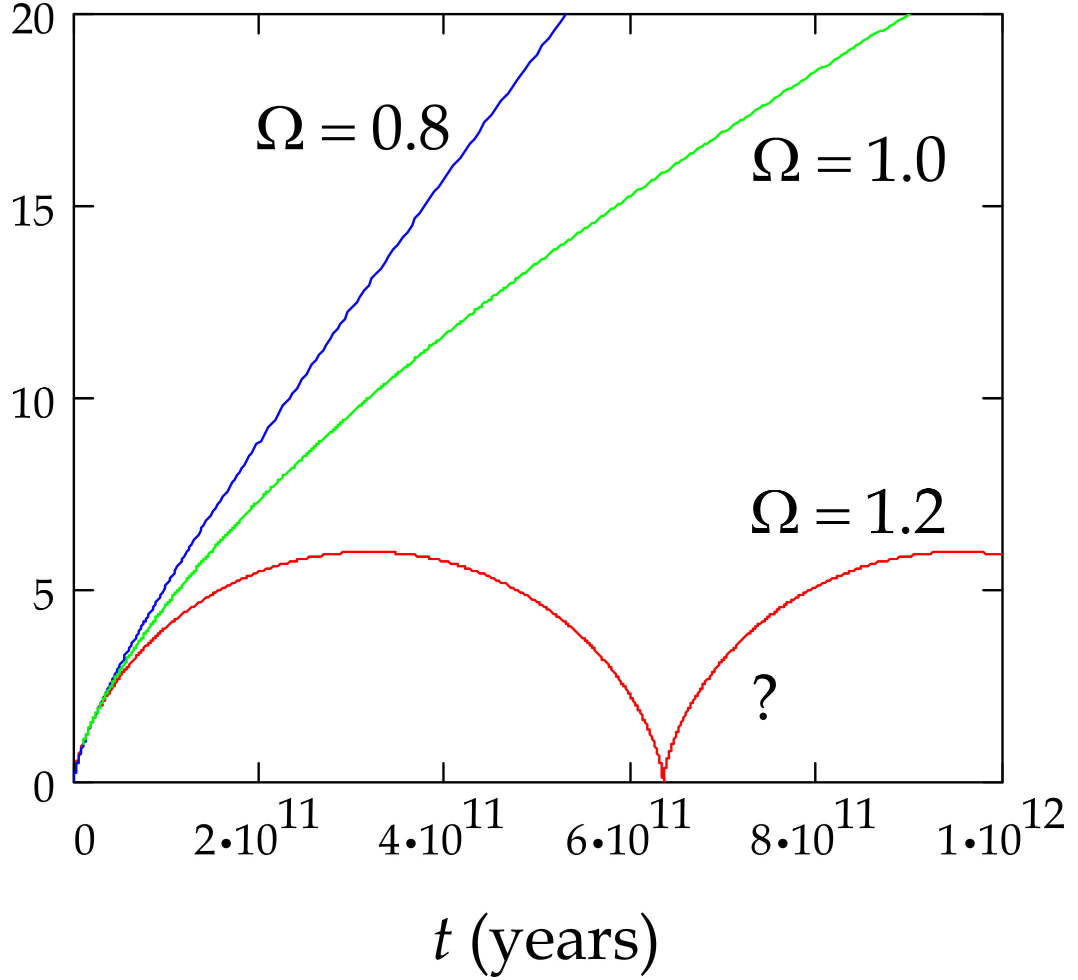
$$D_* = \frac{r}{r_0} = \frac{\Omega_0}{2|1-\Omega_0|}(\cosh\eta - 1)$$

$$t = \frac{\Omega_0}{2H_0|1-\Omega_0|^{3/2}}(\sinh\eta - \eta)$$

Remember, though: $\Omega_0 < 1$ for an open universe, $\Omega_0 > 1$ for a closed one. Plot *D vs. t* by computing them over some range of η (0-10 in the following).

How old is the Universe? (continued)

$$D(t) = \frac{R(t)}{R(t_0)}$$



How old is the Universe? (continued)

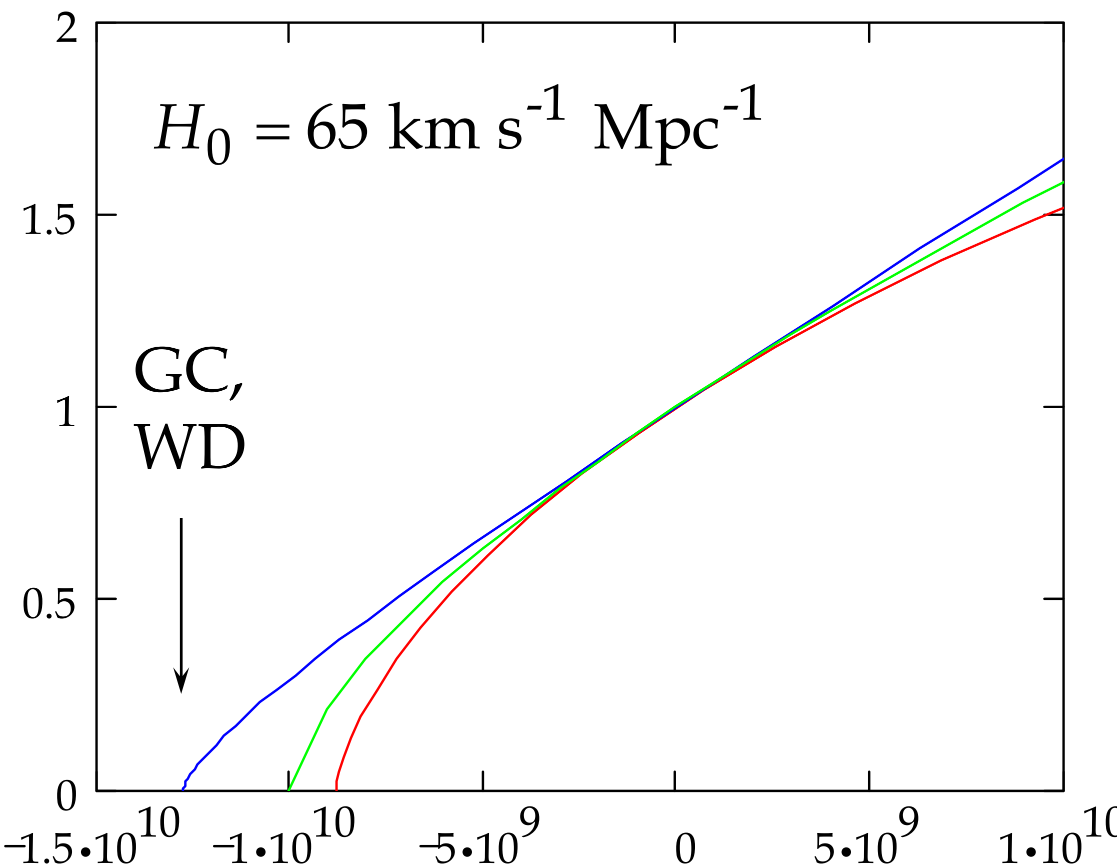
In next week's Homework (#10), you will be asked to show, for a closed matter-dominated universe, that the maximum value of *D*, and the time interval between singularities, are

$$D_{\mathrm{max}} = \frac{R_{\mathrm{max}}}{R(t_0)} = \frac{\Omega_0}{|1-\Omega_0|}$$

$$t_{\text{bang-crunch}} = \frac{\Omega_0}{H_0\,|1-\Omega_0|^{3/2}} \quad .$$

How old is the Universe? (concluded)

Models with various Ω that pass through $D = 1$ at present:



The arrow marks the age of the oldest globular clusters and white dwarfs in the Milky Way. Thus: **if it's matter-dominated, the U is about 13 billion years old.**