Astronomy 142 Recitation #2

25 January 2013

Formulas to remember

The notes for this week’s lectures are full of definitions and you should have the whole lot at your fingertips eventually. Here are some of them, that might be useful today.

**Trigonometric parallax**

\[ r = 1\text{AU}/\tan p \equiv 1\text{AU}/p \quad (p \text{ in radians}) \]
\[ r (p = 1\text{arcsec}) = 3.086 \times 10^{18}\text{cm} = 3.262 \text{light-years} \equiv 1\text{parsec} \]
\[ r = (1\text{parsec})/|p|\text{arcsec} \]

**Magnitudes and color indices** (here all the ms and Ms are magnitudes)

Flux, distance and luminosity:

\[ f = L/4\pi r^2 . \]

Apparent magnitude and flux:

\[ m_2 - m_1 = 2.5\log \left( \frac{f_1}{f_2} \right) \]

This applies to any kind of magnitude, if the fluxes are measured in the same band as the corresponding magnitudes.

Absolute magnitude (= apparent magnitude for an object 10 parsecs away):

At any wavelength: \[ M_\lambda = m_\lambda - 5\log (r/10 \text{pc}) \]
Bolometric: \[ M = m - 5\log (r/10 \text{pc}) - 4.75 - 2.5\log \left( \frac{L}{L_\odot} \right) \]
Distance modulus: \[ DM = m - M = 5\log (r/10 \text{pc}) \]

Color index: \[ CI (\lambda_1, \lambda_2) = m_{\lambda_1} - m_{\lambda_2} = M_{\lambda_1} - M_{\lambda_2} = 2.5\log \left( \frac{f (\lambda_2)}{f (\lambda_1)} \right) \]
\[ B - V = m_B - m_V = M_B - M_V = 2.5\log \left( \frac{f (V)}{f (B)} \right) \]

\[ B - V \approx -0.93 + 9000 \, K/T_e \]
(see FA, p. 317, and also Homework #1)

Bolometric correction (get bolometric magnitude from \( V \) magnitude; see Figure 1):

\[ m = m_V + BC \quad M = M_V + BC \]

Note that magnitudes are **dimensionless**.

**Common (base 10) and natural (base e) logarithms:**

\[ \log x = \frac{1}{\ln 10} \ln x = \log e \ln x = 0.434 \ln x \]

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By "log" we mean common logarithm, and by "ln" natural logarithm.

Planck blackbody function (power per unit area, bandwidth and solid angle):

\[ B_\lambda (\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}; \quad \text{peak at } \lambda_{\text{max}} T = 0.29 \text{ cm K} \].

Note that our main textbook (FA) calls the Planck function \( I_\lambda (\lambda, T) \). This is an eccentric choice of symbols; the rest of the Universe uses \( B_\lambda (\lambda, T) \) and so shall we. The flux from a blackbody, emitted within a small bandwidth \( \Delta \lambda \) and a small solid angle \( \Delta \Omega \) :

\[ f = B_\lambda (\lambda, T) \Delta \lambda \Delta \Omega \].

Solid angle

\[ d\Omega = \sin \theta d\theta d\phi \quad \Omega = \int \int \sin \theta d\theta d\phi \]

\[ \Omega \cong \pi \Delta \theta^2 \] for a cone with small angular radius \( \Delta \theta \).

Binary stars and their motions (here all the ms and Ms are masses; \( a \) is an orbital semimajor axis length; \( a = r \) for circular orbits)

Doppler effect:

\[ \frac{\lambda_{\text{observed}} - \lambda_{\text{rest}}}{\lambda_{\text{rest}}} = \frac{v_r}{c} \]

Center of mass, circular orbits:

\[ \frac{m_1}{m_2} = \frac{r_2}{r_1} \]

Kepler’s third law, any eccentricity:

\[ P^2 = \frac{4\pi^2}{G(m_1 + m_2)} (a_1 + a_2)^3 \]

Kepler’s third law, circular orbits:

\[ P^2 = \frac{4\pi^2}{G(m_1 + m_2)} (r_1 + r_2)^3 \] (see HW#1)

Conservation of momentum:

\[ \frac{m_1}{m_2} = \frac{a_2}{a_1} = \frac{v_2}{v_1} = \frac{v_2}{v_1} \]

Mass function (see HW#1):

\[ f(m_1, m_2) = \frac{P}{2\pi G} v_1^3 < m_2 \]
**Workshop problems**

**Warning!** The workshop problems you will do in groups in Recitation are a crucial part of the process of building up your command of the concepts important in AST 142 and subsequent courses. Do not, therefore, do your work on scratch paper and discard it. Better for each of you to keep your own account of each problem, in some sort of bound notebook.

1. **Magnitudes and distances**
   
a. The bright star Vega (α Lyrae; brightest star in the northern celestial hemisphere) has apparent magnitude 0.0 at all the usual visible wavelengths (U,B,V). What would its apparent magnitude at these wavelengths be, if Vega were moved a factor of five further away?

b. Vega’s parallax is 0.129 arcsec. What is its absolute V magnitude?

c. The Pleiades are a cluster of relatively young stars in the constellation Taurus. (The stars in a cluster that occupies a very small patch of sky, like the Pleiades, can all be assumed to lie approximately the same distance away from us.) Measurements of their apparent V magnitudes and B-V color indices appear in Figure 2. Given your results for Vega, how far away are the Pleiades?

d. What are the absolute V magnitudes of the bluest (smallest B-V) and reddest (largest B-V) stars in the Pleiades?

e. What are the absolute bolometric magnitudes of the bluest and reddest stars in the Pleiades? (You will need to use the chart of bolometric correction given in Figure 1 and Tuesday’s lecture notes.)

2. Suppose all binary stars consisted of two stars with radius $R_\odot$ orbiting each other with constant separation 1 AU, like the example shown in lecture on Tuesday.

   a. From the viewpoint of one of these systems: what is the solid angle which, if observers lay within it, they could see the system eclipse?  *(Hint: recall the trig identity $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$.)

   b. What is the solid angle of the whole sky?

c. What, therefore, is the fraction of binary stars in which we could detect eclipses? Compare this result to that noted in the two largest unbiased surveys that could detect binaries, those by the Kepler and Hipparcos satellites: 1.2% and 0.8% respectively *(Prsa et al. 2010)*.

3. **Binary stars**

   a. Armed with nothing but Newton’s second law, prove Kepler’s third law for bodies in circular orbit about a much-more-massive object with mass $M$. 

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Figure 2: apparent V magnitude as a function of color index B-V for the Pleiades *(Stauffer et al. 1994)*.
b. Armed with nothing but Newton’s second law, derive the form of Kepler’s third law for two objects with equal mass \( m \) in circular orbits about their common center of mass.

Learn your way around the sky, lesson 2. (An exclusive feature of AST 142 recitations.) You may find the lab’s celestial globes, and TheSky running on the lab computers, useful in answering these questions about the celestial sphere and the constellations.

4. a. A good illustration of how first- and second-magnitude stars look is the constellation Ursa Major. How many first-magnitude \((V = m_V < 2, \geq 1)\) and second-magnitude \((V = m_V < 3, \geq 2)\) are there in Ursa Major?

b. Using the first-magnitude stars in Ursa Major, tell me how to find the stars Polaris (\(\alpha\) Ursae Minoris\(^1\), the bright star closest to the north celestial pole) and Arcturus (\(\alpha\) Boötes).

c. How far from the north celestial pole is Polaris? Give your answer in degrees, and also in units of the angular diameter of the Moon.

g. Calculate the angles (in degrees) between \(\alpha\) and \(\beta\) UMa, and \(\alpha\) and \(\eta\) UMa, using angular-distance equation you derived last week.

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\(^1\) Per tradition, we use the Latin names of the constellations. Unfortunately the tradition has also come down to us to treat the names as Latin nouns and to give them their proper declension. Thus they appear in accusative form above (e.g. Ursa Major), except when used as a Greek stellar name, in which case the genitive form is used (e.g. \(\alpha\) Ursae Majoris = “the brightest star of Ursa Major”).
Solutions

1. a. Flux decreases with the square of distance from the star:

\[ f = \frac{L}{4\pi r^2} . \]

We’re talking about the same \( L \) at two different distances; let’s call the near one \( r_2 \) and the further one \( r_1 = 5r_2 \). Then,

\[
m_1 = m_2 + 2.5\log \left( \frac{f_2}{f_1} \right) = m_2 + 2.5\log \left( \frac{r_1^2}{r_2^2} \right) = m_2 + 5\log \left( \frac{r_1}{r_2} \right) = 0 + 5\log 5 = 3.5 .
\]

b. Vega’s distance, from its parallax, is \( r = 1 \) parsec/0.129 arcsec = 7.75 parsecs, so

\[
M_V = m_V - 5\log (r/10 \text{ pc}) = 6.6 - 5\log (7.75/10) = 0.55
\]

c. We may assume that the Pleiades with \( B - V = 0 \) are just like Vega but lie at a different distance. Reading the \( V \) magnitudes of these stars off of Figure 2, we get \( V = m_V \equiv 6.6 \). Calling the Pleiades 1 and Vega 2,

\[
m_1 - m_2 = 2.5\log \left( \frac{f_2}{f_1} \right) = 5\log \left( \frac{r_1}{r_2} \right)
\]

\[
r_1 = r_2 10^{(m_1-m_2)/5} = 7.75 \text{ parsecs} \times 10^{(6.6-0)/5} = 160 \text{ parsecs}.
\]

In other words, the distance modulus of the Pleiades is \( DM = m_V - M_V = 6.6 - 0.55 = 6.0 \). All the Pleiades appear fainter than their absolute magnitudes by about 6 magnitudes.

(This distance is close to the best measurements, but not quite on the nose. The Hipparcos team measured the trig parallaxes of many Pleiades and got a distance of 135 parsecs, once they got their systematic errors under control.)

d. The bluest Pleiades have \( V \equiv 2.7 \) and the reddest have \( V \equiv 15.5 \). Since their distance modulus is about 6.0 magnitudes, their absolute \( V \) magnitudes are \( M_V = -3.3 \) and \( 9.5 \) respectively.

e. The bluest Pleiades have \( B - V \equiv -0.1 \) and the reddest have \( B - V \equiv 1.5 \). From Figure 1 we get \( BC = -0.86 \) and \(-1.79 \) respectively, so the bolometric magnitudes are \( M = M_V + BC = -4.2 \) and \( 7.7 \) respectively.

2. a. In class we showed that there would be a “grazing” eclipse if the orbital plane were inclined with respect to the line of sight by \( 2R_\odot/\text{AU} \). Clearly there would also be a grazing eclipse for a tilt of \(-2R_\odot/\text{AU} \). So, in terms of the inclination angle with respect to an observer’s line of sight, \( i \), an eclipse corresponds to the range \( i = \pi/2 \pm 2R_\odot/\text{AU} \). Thus observers who see the system eclipse will occupy the solid angle
Now use that identity, note that \( \cos(\pi/2) = 0 \) and \( \sin(\pi/2) = 1 \), and use the small angle approximation whenever appropriate:

\[
\Omega = 2\pi \left[ \cos(\pi/2 - 2R_\odot /AU) - 4\pi \sin(2R_\odot /AU) \right] \\
\simeq 8\pi R_\odot /AU = 0.12 \text{ steradians.}
\]

b. Shown in class on Tuesday: \( \Omega = 4\pi \) steradians for the whole sky.

c. The fraction of observers – considered uniformly and randomly distributed over the sky – who can see the eclipse is the same as the ratio of these two solid angles:

\[
F = \frac{8\pi R_\odot}{4\pi AU} = 0.0093 = 0.93\% .
\]

This, of course is the same fraction of eclipses that would be seen by one observer among a randomly-oriented collection of binary stars. The result is quite similar to what \textit{Kepler} and \textit{Hipparcos} actually got.

3. All you need is \( F = ma \).

a. For example,

\[
F = -\frac{GMm}{r^2} = ma = -\frac{mv^2}{r} = -\frac{4\pi^2 mr^2}{p^2} \\
p^2 = \frac{4\pi^2}{GM} r^3 , \text{ q.e.d.}
\]

b. Each will orbit the center of mass, which is halfway between them:

\[
F = -\frac{Gm_1^2}{(2r)^2} = ma = -\frac{v_1^2}{r} = -\frac{4\pi^2 m_1 r_1}{p^2} \\
p^2 = \frac{4\pi^2}{Gm_1 r_1} (2r_1)^2 \left\{ = \frac{4\pi^2}{G(m_1 + m_2)} (r_1 + r_2)^3 , \text{ q.e.d.} \right\}.
\]

3. The three constellations that concern us this week are usually abbreviated in the Greek names of stars, as UMa, UMi and Boo.

a. Three of each. Together with one third-magnitude star (\( \delta \) UMa), and a fourth-magnitude one right next to \( \zeta \) UMa, they are better known as the Big Dipper. Note that the Greek names for the stars do not really indicate the rank-order of brightness.
b. Follow the line through the two stars furthest from the Dipper’s handle (β and α UMa) to Polaris. Follow the curve of the Dipper’s handle (δ through η UMa) to find the bright, orange star Arcturus.

c. The angular diameter of the Moon (and the Sun) is about half a degree. The declination of Polaris, in J2000 coordinates, is $+89^\circ 15' 50.794'' = 89.264^\circ$, so it lies $0.736^\circ$, or about 1.5 moon diameters, from the Pole.

g. I get $5.374^\circ$ and $25.709^\circ$. (Note that this is the same answer that TheSky gives: look near the bottom of the Object Information window.)