Astronomy 142 Recitation #3

1 February 2013

Formulas to remember

Hydrostatic equilibrium

\[ M(r) = \int_0^r \rho(r') 4\pi r'^2 dr' \]
\[ \frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \]
\[ P(R) - P(r) = \int_r^R \frac{dP}{dr'} dr' = -\int_r^R \frac{GM(r')\rho(r')}{r'^2} dr' \]

Scaling: \( \rho_C \propto \frac{M}{R^3}, P_C \propto \frac{GM^2}{R^4}, T_C \propto \frac{GM\mu}{R} \)

Ideal gases

\[ P = \frac{\rho k T}{\mu} \]

where \( \mu \) is the average mass of particles (i.e. atoms or ions or molecules) in the gas.

3-D random walk

To travel a distance \( R \) in randomly-directed steps of length \( \ell \) requires going \( N \) steps, given by

\[ N = 3R^2 / \ell^2 \]

Starlight heating by a variable-luminosity star

If the average luminosity of a star is \( L \) and the average temperature to which this heats a planet is \( T \), then a small variation by \( \Delta L < L \) in the star’s luminosity leads to a small variation \( \Delta T \) in the planet’s temperature, given by

\[ \frac{\Delta T}{T} = \frac{1}{4} \frac{\Delta L}{L} \]

Workshop problems

Warning! The workshop problems you will do in groups in Recitation are a crucial part of the process of building up your command of the concepts important in AST 142 and subsequent courses. Do not, therefore, do your work on scratch paper and discard it. Better for each of you to keep your own account of each problem, in some sort of bound notebook.

1. A star with total mass \( M \) has density given by
\[ \rho(r) = \rho_0 e^{-r/R} . \]

The star doesn’t have a sharp outer edge; \( R \) is just the characteristic radius over which the density changes by a factor of 1/e.

What is the central density \( \rho_0 \), in terms of \( M \) and the characteristic radius \( R \)? (Note: the answer to this question is contained in the answer to the next question, so you can proceed to the next question and come back later with your result for this one if you’d like.)

2. For the star in Problem 1, what is the mass \( M(r) \) contained within a sphere of radius \( r \), in terms of \( r \), the total mass \( M \) and the characteristic radius \( R \)? (Hint: you will have to use integration by parts. It will be convenient, as you may have found in Problem 1, to change variables to \( x = r/R \) early in your solution.) Check your answer for \( r \to \infty \) against your answer to Problem 1.

3. Stellar structure gets complicated in a big hurry. Here’s why the stars you will consider for the rest of the semester will be assumed to density with functional form even simpler than that above.

a. Using your results for Problems 1 and 2, and the hydrostatic-equilibrium equation, derive an expression for the pressure \( P(r) \) at radius \( r \) within the star in Problem 1. Since you have already done enough integration by parts for one day I will hereby give you some of the integrals:

\[
\int e^{ax}/x^2 \, dx = -e^{ax}/x + a \int e^{ax}/x \, dx
\]

\[
\int_0^\infty e^{-x}/x \, dx = \text{Ei}(z) \quad \text{(the Exponential Integral)}
\]

b. Consider the behavior of this result at the center of the star. Can a star with density given as in Problem 1 be in hydrostatic equilibrium?

4. What is the total gravitational potential energy \( U \) for a uniform-density star with mass \( M \) and radius \( R \), in terms of \( M \) and \( R \)?

5. This doesn't specifically have anything to do with the Sun-Earth connection, but since it's recent and emblematic of frequent articles in the popular press:

Citing the HadCRUT combined database of oceanic and land-surface air temperatures, the Daily Mail (UK) claimed in an article published on 13 October 2012 that the UK Meteorological Office “quietly released” data which prove that no systematic change in global temperatures had occurred since

\[ \text{Ei}(z) = \ln z + \sum_{j=1}^\infty \frac{z^j}{j \cdot j!} . \]

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1 The Exponential Integral is an object you’ll learn more about in Math 282. For real positive values of \( z \) it is given by the following expression, obtainable by expanding the exponential integrand in a power series and integrating directly:
1997, and this indicates that global warming has stopped. Here is their plot of the 1997-2012 HadCRUT data:

The Mail claims furthermore that

- this shows global warming to have lasted only from 1980 to 1996; before 1980 global temperatures had been stable or declining since 1940.
- since atmospheric CO₂ continued to increase steadily through this "plateau", that it must have nothing to do with global warming.
- this radical result has reignited the debate among climate scientists about global warming and CO₂, with prominent scientists lining up on both sides.
- the release of these damning data without any notice or fanfare indicates that certain "liberal" climate scientists are covering up evidence against their claims of connection between temperature and greenhouse gases.

Assess these five claims, considering the experiments we discussed in connection with the possibility that Solar variation is behind global climate variation. Which of them are true, in whole or in part?

Learn your way around the sky, lesson 3. (An exclusive feature of AST 142 recitations.) You may find the lab’s celestial globes, and TheSky running on the lab computers, useful in answering these questions about the celestial sphere and the constellations. Today’s lesson is on

6. Sidereal time and hour angle – which will be a review, for AST 111 victims.

   a. A day is measured from noon to noon: that is, the time it takes to return the Sun to its maximum elevation in the sky. Assume that Earth’s orbit is circular; and calculate the angle η (in radians) that Earth’s orbital radius sweeps out in a day.
b. By what angle, in radians, does Earth turn on its axis in the course of a day?

c. Earth spins at a constant rate. How long does it take, in hours, minutes and seconds, for Earth to make one complete rotation? This duration is called the \textit{sidereal day}; it is the time it takes to return the fixed stars to their positions in the sky.

\textit{Sidereal time} tells us where the stars are in the sky, in the same way as ordinary civil time tells us where the Sun is in the sky. The rules are that the sidereal time is zero at the moment of the vernal equinox: the time the Sun passes the celestial equator going north. The vernal equinox is also the origin of right ascension, so the sidereal time is also the right ascension of the stars that lie along the \textit{meridian}: the line of celestial longitude connecting the poles and the \textit{zenith} (the point directly overhead).

d. Suppose the vernal equinox happened precisely at noon on March 21st. What is the right ascension of the star directly overhead precisely at midnight on March 21st? What is its declination?

e. During the same year as in part d: what is the sidereal time at midnight on September 21st?

f. The coordinates of the black hole at the center of the Milky Way Galaxy, an object called Sagittarius A* (Sgr A*), are $\alpha = 17^h \ 45^m \ 40.0^s$, $\delta = -29^\circ \ 00' \ 28''$. On what date is Sgr A* on the meridian at midnight? (That is, on what date is the sidereal time at midnight equal to the object’s right ascension?) What would be a good time of year to observe Sgr A*?
Solutions

1. It’s probably easier to do 2 if you practice on 1 first:

\[
M = \int \rho dV = 4\pi \rho_0 \int_0^\infty r^2 e^{-r/R} dr' = 4\pi \rho_0 R^3 \int_0^\infty x^2 e^{-x} dx
\]

\[
= 4\pi \rho_0 R^3 \left( -x^2 e^{-x} \bigg|_0^\infty + 2 \int_0^\infty xe^{-x} dx \right)
\]

\[
= 4\pi \rho_0 R^3 \left( -2xe^{-x} \bigg|_0^\infty + 2 \int_0^\infty e^{-x} dx \right)
\]

\[
= 4\pi \rho_0 R^3 \left( -2e^{-x} \bigg|_0^\infty \right)
\]

\[
= 8\pi \rho_0 R^3 ;
\]

\[
\rho_0 = \frac{M}{8\pi R^3} .
\]

2. Hope it doesn’t get confusing to use \( M \) and \( M(r) \) to mean different things in the same equation:

\[
M(r) = \int \rho dV = 4\pi \rho_0 \int_0^r r'^2 e^{-r'/R} dr' = 4\pi \rho_0 R^3 \int_0^{r/R} x^2 e^{-x} dx
\]

\[
= \frac{M}{2} \left( -x^2 e^{-x} \bigg|_0^{r/R} + 2 \int_0^{r/R} xe^{-x} dx \right)
\]

\[
= \frac{M}{2} \left( -\left( \frac{r}{R} \right)^2 e^{-r/R} - 2xe^{-r/R} \bigg|_0^{r/R} + 2 \int_0^{r/R} e^{-x} dx \right)
\]

\[
= \frac{M}{2} \left( -\left( \frac{r}{R} \right)^2 e^{-r/R} - 2\frac{r}{R} e^{-r/R} - 2e^{-r/R} \bigg|_0^{r/R} \right)
\]

\[
= \frac{M}{2} \left( 2 - \left( \frac{r}{R} \right)^2 + 2\frac{r}{R} + 2 e^{-r/R} \right)
\]

3. a. Just plug the last result into the integral form of the hydrostatic-equilibrium equation and integrate away:

\[
P(x) - P(r) = \int_r^\infty \frac{dP}{dr'} dr' = -G \int_r^\infty \frac{M(r') \rho(r')}{r'^2} dr'
\]

that is, after noting that the boundary condition is \( P(x) = 0 \):
\[ P(r) = -\frac{GM^2}{16\pi R^3} \int r \left( 2 - \left( \frac{r'}{R} \right)^2 + 2\frac{r'}{R} + 2 \right) e^{-r'/R} \frac{e^{-r/R}}{r'^2} dr' \]

\[ = -\frac{GM^2}{16\pi R^3} \int \left( 2 - \left( x^2 + 2x + 2 \right) e^{-x} \right) \frac{e^{-x}}{x^2} dx \]

\[ = -\frac{GM^2}{16\pi R^4} \int \left( 2e^{-x} - e^{-2x} - 2e^{-2x} - 2 \frac{e^{-2x}}{x^2} \right) dx \]

\[ = -\frac{GM^2}{16\pi R^4} \left[ -2 \frac{e^{-x}}{x} |_{r/R}^\infty - 2 \int \frac{e^{-x}}{x} dx + \frac{e^{-2x}}{2} |_{r/R}^\infty \right. \]

\[ \left. - 2 \int \frac{e^{-2x}}{x} dx + 4 \frac{e^{-2x}}{x} |_{r/R}^\infty \right. \]

\[ = -\frac{GM^2}{16\pi R^4} \left[ \frac{R}{r} e^{-2r/R} - \frac{2R}{r} e^{-r/R} + \frac{1}{2} e^{-2r/R} + 2Ei \left( \frac{r}{R} \right) + 2Ei \left( \frac{2r}{R} \right) \right] \]

b. The exponentials are all finite at all \( r \), but the terms with \( r \) in the denominator, and the first term in the expansion of the Exponential Integral, all blow up at \( r = 0 \). Thus there’s no way the density distribution can satisfy the hydrostatic equilibrium equation at \( r = 0 \); it can’t be a hydrostatic configuration.

4. This is just a reminder in how to compute gravitational potential energies of round objects: imagine first bringing an infinitesimal mass \( dm \) in from infinity to a mass \( M(r) \).

\[ dU = -W = + \int dF(r') dr' = \int \frac{Gdm}{r'^2} M(r) \frac{r^3}{R^3} dr' \]

\[ = \frac{GMdm}{R^3} r^3 \int_0^1 \frac{r'^3}{r'^2} = \frac{GMdm}{R^3} \left[ -\frac{1}{r'} \right]_0^1 = -\frac{GMdm}{R^3} r^2 \]

Then let \( dm = \rho dV = \left( \frac{3M}{4\pi R^3} \right) 4\pi r^2 dr \), so that

\[ U = \int dU = -\frac{3GM^2}{R^6} \int_0^R r^4 dr = -\frac{3GM^2}{5} \]

5. A good and complete answer is given by Phil Plait in his Bad Astronomy blog, which now appears on Slate. The Mail is a lot like the New York Post. Or maybe I should put it the other way around; this brand of tabloid publication was really invented in Britain. The claims, one by one:

- To find a plateau requires some extreme cherry-picking of the data: that is, the ignoring of other perfectly-good parts of the data which exhibit a trend. The particular form of cherry-picking in which they indulged involved going back in the (noisy) monthly temperature data until they found a month with the same temperature as currently (which turns out to be October 1997) and trimming the plot accordingly. Here are the monthly northern-hemisphere HadCRUT4 data from
1970 to present (blue), the Mail’s cropping (red), and a linear fit to all the data (dashed green):

So that claim is phony; they used the noise in the data to make it look like a plateau.

- Looks to me, from the plot above, as if the warming trend started earlier than 1980, so this claim is false too. By the way, since HadCRUT has land data in it, it is fundamentally noisier than the ocean surface data, simply because the oceans (which convect and conduct heat better than rocks and dirt) couple a larger thermal reservoir to the air than land does. This is why we prefer to model the ocean temperatures. Here are the northern-hemisphere MOST (green) and HadCRUT (blue), compared:

- So there is still a warming trend to go with the CO₂ trend and their third claim is also false.

- They illustrate their newfound scientific controversy by opposing the views of Phil Jones to those of Judith Curry. Dr. Jones is an eminent and respected climate scientist who, as a leading member of the IPCC, shared a Nobel Prize for his work; he represents a consensus which involves better
than 99% of scientists. Prof. Curry is a convinced Denier who, when the initially-skeptical BEST group produced a reanalysis of the temperature data which agreed with the IPCC results, left that group and denounced them on Fox News. They don't belong in the same debate; only a few cranks are left on the climate-change denial side, and it takes more evenly balanced sides to make a controversy. (By the way, Georgia Tech, where Judith Curry works, doesn't have a "climate science department", but she is Chair of their Earth and Atmospheric Sciences department.)

- The folks who run HadCRUT update their database in real time. There isn't a press release for every addition to the data. I don't call a press conference every time I publish a paper; does this make me guilty of suppressing my data?

So the score is five claims, none true. The usual, for publications of the Mail's quality.

6. a. The year is 365.25 days long, so every day it goes through $1/365.25$ of its orbit. One full orbit is $2\pi$ radians, so the angle covered in a day is just

$$\eta = \frac{2\pi}{365.25} = 0.0172023$$

b. The Earth rotates in the same direction that it revolves around the Sun. It has to turn $\eta$ more than a full turn every day, in order to get the Sun back to noon.

c. Thus

$$\frac{\Delta t}{2\pi} = \frac{1}{2\pi + \eta}$$

$$\Delta t = \frac{2\pi}{2\pi + \eta} 1 \text{ day} = 0.99727 \text{ days} = 23^{\text{h}} 56^{\text{m}} 04.1^{\text{s}}$$

is the length of the sidereal day. The sidereal day is shorter than the solar one: the stars get back to their former positions 3 minutes and 56 seconds faster than the Sun does. Thus the stars also rise earlier, and set earlier, every night, according to our solar clocks. (As you've probably noticed.)

d. The twelve hours of the day is half a solar day, during which the Earth turns $\pi + \eta/2$ radians. At the beginning of this period the RA of stars straight overhead was zero, and now therefore it's $\pi + \eta/2 = 12^{\text{h}} 01^{\text{m}} 58.3^{\text{s}}$. Since it's straight overhead its declination is equal to Rochester's latitude, $+43^\circ 10' 00''$

e. Should be close to zero, since it's about six months later. In detail: that's 184 days, or 0.50376 years. In 1 year, sidereal time advances 24 hours, so in this case it advances 12.09 hours = $12^{\text{h}} 05^{\text{m}} 25.3^{\text{s}}$. Adding this to the answer of part d, we get $24^{\text{h}} 07^{\text{m}} 23.6^{\text{s}} = 0^{\text{h}} 07^{\text{m}} 23.6^{\text{s}}$.

f. Well, let's see: ST at midnight on March 21 is $\pi + \eta/2 = 12^{\text{h}} 03^{\text{h}}$. The right ascension of Sgr A* is $17.76^\text{h}$, $5.73^\text{h}$ later than the midnight meridian on March 21st. The sidereal time at midnight goes through 24 hours in a year, so the fraction of the year elapsed between midnight March 21st and the midnight that Sgr A* is $5.73/24 = 0.24$, which is about 87 days. Thus Sgr A* is on the meridian at midnight around 16 June. Within a month or two of 16 June would seem to be a good time to observe Sgr A*.