Astronomy 142 Recitation #12

19 April 2013

Formulas to remember

Observed temperature of a distant blackbody:
\[ T_0 = \frac{TR(t)}{R(t_0)} = \frac{T}{1+z} \]

Robertson-Walker absolute interval
\[
d s^2 = c^2 dt^2 - R^2(t) \left[ \frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]
\]

Friedmann equation
\[
\left( \frac{\dot{R}}{R} \right)^2 - \frac{8\pi G}{3} \rho_m - \frac{\Lambda}{3} = -\frac{K c^2}{R^2}
\]

Scale factor
\[ R r_e = r \]

Critical density (present value)
\[ \rho_c = \frac{3c^2 H_0^2}{8\pi G} = 9.3 \times 10^{-9} \text{ erg cm}^{-3} \]

Workshop problems

Warning! The workshop problems you will do in groups in Recitation are a crucial part of the process of building up your command of the concepts important in AST 142 and subsequent courses. Do not, therefore, do your work on scratch paper and discard it. Better for each of you to keep your own account of each problem, in some sort of bound notebook.

1. Return to the galaxy studied in problem 4 on homework #7. Suppose that the dark halo follows its given functional form out to a radius, \( R_1 \), that is ten times the scale radius \( R_0 \), after which it drops more rapidly and becomes negligible. Integrate the expressions for mass and light per unit area from the center out to \( R = R_1 = 10R_0 \), use the expression derived for \( \rho_0 / L(0) \), and obtain thereby a value for \( M/L \) that applies to the whole galaxy. What is this value, in solar units? We will use it in the next problem as a \( M/L \) ratio typical of galaxies.

2. Do the calculations for which the answers were given in class on 11 April: The Coma cluster contains about 10000 galaxies, has a core radius \( r_0 \) of about 3 Mpc, and a radial-velocity dispersion \( \sigma_v \) of 977 km sec\(^{-1} \). Typically the luminosity of the cluster galaxies is \( L_S = 5 \times 10^8 L_\odot \). The cluster also contains \( M_{X-ray} = 3 \times 10^{14} M_\odot \) in gas at temperature \( T = 10^8 \text{ K} \)

   a. Calculate the mass in galaxies (using the results of problem 1) and the virial mass for the Coma cluster, and compare these results to the mass of X-ray-emitting gas. What fraction of the cluster’s mass is dark matter?
b. Calculate the crossing time and the relaxation time for the cluster, and compare these with the age of the Universe, 14 billion years. Has the cluster had time to come to virial equilibrium in the usual manner, by numerous elastic collisions of the galaxies?

c. Calculate the thermal speed of hydrogen in the hot X-ray emitting gas, and the escape speed from the sum of the galaxy and X-ray masses, and also from the virial mass. Is the presence of hot X-ray-emitting gas a strong argument for the presence of substantial dark matter, on its own?

3. **What came before the Big Bang?** We know so far of three different forms of the absolute interval between events: that for flat spacetime (special relativity), which in Cartesian form is

\[ ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \; ; \]

that for the surroundings of the horizon of a nonspinning black hole of mass \( M \), which in the spherical coordinates of a distant observer is

\[ ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{2GM}{rc^2}} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \; ; \]

and that corresponding to the Robertson-Walker metric, which in stretchy-spherical-grid coordinates is

\[ ds^2 = c^2 dt^2 - R^2 \left( \frac{dr^2}{1 - Kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) . \]

a. Consider two events consisting of ticks on a single, given, clock. What is the algebraic sign of \( ds^2 \) between these two events? (This would be called a **timelike** interval.)

b. Consider two events consisting of positions in space with separation measured by a ruler. What is the algebraic sign of \( ds^2 \) between these two events? (This would be called a **spacelike** interval.)

c. The form of the absolute interval in the vicinity of a mass-density singularity is very complicated, but has one particularly striking property: \( ds^2 < 0 \) under all conditions. (Make sure you see this demonstrated some day, in a GR class, or by reading the works of Belinskii, Khalatnikov and Lifshitz, for example in *Adv. Phys.* 31, 639-667 [1982].) Is there such a thing as timelike interval near a singularity? Is there such a thing as time, as we know it, near a mass-density singularity?

d. What happened before the Big Bang? Discuss.

4. **No math involved.** Consider a universe in which each distant galaxy we can see has been traveling at the same speed with respect to us since the beginning of time. Show that the present age of this universe is

\[ t = \frac{1}{H_0} , \]

where \( H_0 \) is the present value of the Hubble constant.
Learn your way around the sky, lesson 11. (An exclusive feature of AST 142 recitations.) Use the lab’s celestial globes, TheSky running on the lab computers, the SIMBAD database at http://simbad.harvard.edu/simbad/, and any other resources you would like to use, to answer these questions about the celestial sphere and the constellations.

5. The winter constellation we have been using as our landmarks are beginning to set; time to learn some springtime ones.

a. Describe how a naked-eye observer could locate the first-magnitude star Spica (α Virginis), using landmarks you already know well.

b. Describe how a naked-eye observer could locate the core of the Virgo cluster, the nearest rich cluster of galaxies and the densest part of the supercluster to which our Galaxy belongs.

c. If you follow the directions in part b, you do indeed see a fuzzy collection of faint objects nearby, with the naked eye. Is this the Virgo cluster?

d. Describe how a naked-eye observer could locate the core of the Coma cluster.

e. What is your favorite Virgo Cluster galaxy?
Solutions

1. First integrate \( L(R) = L(0) e^{-(R/R_0)} \) over radius:

\[
L = 2\pi \int_0^{R_1} e^{-(R/R_0)} RdR = 2\pi R_0^2 L(0) \int_0^{R_1/R_0} u e^{-u} du
\]

\[
= 2\pi R_0^2 L(0) \left[ u e^{-u} \bigg|_0^{R_1/R_0} + \int_0^{R_1/R_0} e^{-u} du \right]
= 2\pi R_0^2 L(0) \left[ \frac{R_1}{R_0} e^{-R_1/R_0} - e^{-R_1/R_0} \right]
= 2\pi R_0^2 L(0) \left[ \frac{R_1}{R_0} + 1 \right] \approx 2\pi R_0^2 L(0)
\]

And then integrate \( \mu(R) = \frac{\pi \rho_0 R^2}{R} \) over radius:

\[
M = 2\pi \rho_0 \int_0^{R_1} dR = 2\pi^2 \rho_0 \frac{R_1^3}{3}
\]

\[
\frac{M}{L} = \frac{2\pi^2 \rho_0 \frac{R_1^3}{3}}{2\pi R_0^2 L(0)} = \pi \rho_0 \frac{R_1^2}{L(0) R_0}
\]

\[
= \frac{1}{\pi^2} \frac{M_\odot}{L_\odot} \ kpc^{-1} \ 20 \ kpc \cong 7.4 \frac{M_\odot}{L_\odot}
\]

2. a. The total luminosity of the galaxies in the cluster is thus \( L_g = 5 \times 10^{12} L_\odot \), and using the mass-to-light ratio we just got that implies a mass in galaxies (including haloes) of \( M_g = 3.7 \times 10^{13} M_\odot \).

The virial mass, as we saw the other day in class, is

\[
M_{\text{virial}} = \frac{6 \rho_0 v_r^2}{G} = 4.0 \times 10^{15} M_\odot
\]

So the masses - virial, hot intergalactic gas, and galaxies - are roughly in the ratio 100:10:1. (These are all the real numbers for Coma, BTW.) The cluster is 90% dark matter.

b. First: the random velocity must be inferred from the dispersion of radial velocity:

\[
\overline{v_r^2} = v^2 \cos^2 \theta = \frac{v^2}{3} \Rightarrow v = \sqrt{3\overline{v_r^2}} = 1700 \text{ km sec}^{-1}
\]

Now one can plug into the formulas in the lecture notes from 17 March:

\[
t_x = \frac{2R}{v} = 3.5 \times 10^9 \text{ year}
\]

\[
t_c = t_x \frac{N}{24\ln(N/2)} = 1.6 \times 10^{11} \text{ year}
\]
The Coma cluster is thus significantly younger than its relaxation time, because the Universe itself is; it has not had time to come to equilibrium by the usual (galaxy-galaxy collisional) means.

c. Thermal and escape speeds are

\[ v_{th} = \sqrt{\frac{3kT}{m_H}} = 1600 \text{ km sec}^{-1} \]
\[ v_{esc} = \sqrt{\frac{2GM}{r_0}} = 1100 \text{ km sec}^{-1} \text{ for } M = M_g + M_{X-ray} \]
\[ = 3400 \text{ km sec}^{-1} \text{ for } M = M_{\text{virial}}. \]

Since the thermal speed would exceed the escape speed if \( M = M_g + M_{X-ray} \), the distribution of gas would not be there any more unless it got there very recently. (The Jeans escape time comes out to \( 7.4 \times 10^9 \) years, and the galaxies in the cluster seem to be practically as old as the Universe, \( 1.4 \times 10^{10} \) years.) But if the mass were the virial mass, the thermal speed would be substantially less than the escape speed, and the hot gas would be solidly bound to the cluster. (The Jeans escape time comes out to \( 4.4 \times 10^{10} \) years.) This is a powerful argument that the virial mass is the right one, that the contents of the cluster are in equilibrium, and that it is 90% dark matter.

3. a. Since the absolute interval is absolute, we can calculate it in any reference frame we want; I choose the clock’s rest frame. On our single, given clock, \( dx^2 = dy^2 = dz^2 = 0 \), and similarly in the spherical coordinate systems, because the two events happen at the same spatial location in the clock’s rest frame. Thus \( ds^2 = c^2 dt^2 > 0 \).

b. This time I’ll choose the reference frame of the person holding the ruler, to whom the interval was measured in an instant: \( dt^2 = 0 \). Thus \( ds^2 = -d\ell^2 < 0 \).

c. Since \( ds^2 < 0 \) under all conditions, there is such a thing as timelike interval near a singularity. Thus all four spacetime coordinates, and the intervals between them, are spacelike near a singularity; there is no such thing as time as we know it.

d. If there is no such thing as time at the Universal singularity, then time began with the Big Bang, and there was no “before.” At least not in this Universe and these four dimensions. (To read a nice discussion of alternatives to this result that arise from ideas that our Universe is only one of a large number of universes - the Multiverse - which inhabit a much larger manifold of dimensions, check out the recent book by our own Prof. Steve Manly.)

4 This of course doesn’t count as math:

\[ d = vt \]
\[ t = \frac{d}{v} = \frac{d}{v_r} \text{ on large scales} \]
\[ = \frac{1}{H_0} = 13.2 \text{ Gyr}. \]

\( 1/H_0 \), called the Hubble time, turns out to be fairly close to all the current, reasonable, estimates of the age of the Universe.
4. a. Follow the curve of the handle of the Big Dipper to Arcturus. Keep following the same curve, as far from Arcturus as Arcturus is from the end of the handle, and you’ll find Spica. See Figure 1.

b. The core of the Virgo cluster makes an equilateral triangle with Arcturus and Spica. Most of it is contained within the slightly taller, nearly isosceles triangle bounded by Arcturus, Spica and the second-magnitude star Denebola (β Leonis). See, again, Figure 1.

c. The individual galaxies are too faint to see with the naked eye – the brightest ones are at least a couple of magnitudes fainter than M3, and one has to struggle to see that with naked eye – and they’re not close enough together to act like nebulosity to the naked eye. The fuzzy cluster of faint things is a stellar cluster: the open cluster Coma Berenices, eponym of the constellation to which it belongs. See, again, Figure 1.

d. The Coma cluster is called that because it’s in Coma Berenices, and thus turns out to be close on the sky to the Virgo cluster, but is about five times further away. From the Virgo cluster, go to the center of the Coma Berenices stellar cluster, turn left and go half again as far. See, one more time, Figure 1.

e. I bet most will say M 87. Mine is M 100.

Figure 1: finding chart for Spica, and the Virgo and Coma Clusters, looking east just after Spica rises (~8 PM).