Astronomy 203 Final Exam Solutions

16 December 1999

Exam rules: You may consult *only* one page of formulas and constants, a calculator, and a computer running (only) RayTrace while taking this test. You may *not* consult any books, nor each other. All of your work must be written on the attached pages, using the reverse sides if necessary, or the accompanying diskette. The final answers, and any formulas you use or derive, must be indicated clearly. Exams are due three hours after we begin, and will be returned Monday, 12/20/99.

Work out any FIVE of the six problems on this exam. Each is worth 40 points, and the maximum score is 200 points. If you do all six, the best five will determine your score.

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1	2	3	4	5	6	Total
40	40	40	40	40	40	240

Problem 1

Starting with the Kirchhoff diffraction integral, calculate the intensity of normally-incident light diffracted by a rectangular aperture with lengths along the x *and* y *directions respectively* a *and* b*, as a function of angles with respect to the optical axis, on a screen in the far field.*

This of course is much like Problem 10 on the Practice Final Exam.

$$\begin{aligned}
E_F\left(\kappa_x,\kappa_y,t\right) &= \frac{e^{i\kappa r}}{\lambda r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_N(x',y',t)e^{-i\left(\kappa_x x'+\kappa_y y'\right)}dx'dy' \\
&= \frac{e^{i(\kappa r-\omega t)}E_0}{\lambda r}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} e^{-i\left(\kappa_x x'+\kappa_y y'\right)}dx'dy' \\
&= \frac{e^{i(\kappa r-\omega t)}E_0}{\lambda r}\left.\frac{e^{-i\kappa_x x'}}{-i\kappa_x}\right|_{-a/2}^{a/2}\left.\frac{e^{-i\kappa_y y'}}{-i\kappa_y}\right|_{-b/2}^{b/2} = \frac{4e^{i(\kappa r-\omega t)}E_0}{\lambda r\kappa_x\kappa_y}\sin\left(\frac{\kappa_x a}{2}\right)\sin\left(\frac{\kappa_y b}{2}\right)
\end{aligned}$$

$$\begin{aligned}
I\left(\kappa_x,\kappa_y\right) &= \frac{c}{8\pi}E_F E_F^* \quad \text{(CGS units)} \\
&= \left(\frac{4E_0}{\lambda r\kappa_x\kappa_y}\right)^2\sin^2\left(\frac{\kappa_x a}{2}\right)\sin^2\left(\frac{\kappa_y b}{2}\right)
\end{aligned}$$

Recall that $\kappa_x = \kappa\theta_x = \kappa x/r$, and similarly for κ_y,where x and y are rectangular coordinates on the screen; then

$$I\left(\theta_x,\theta_y\right) = ab\left(\frac{E_0}{\lambda r}\right)^2\left(\frac{\sin\left(\frac{\kappa a\theta_x}{2}\right)}{\frac{\kappa a\theta_x}{2}}\right)^2\left(\frac{\sin\left(\frac{\kappa b\theta_y}{2}\right)}{\frac{\kappa b\theta_y}{2}}\right)^2 \quad .$$

Furthermore, note that

$$\lim_{\theta_x\to 0}\frac{\sin\left(\frac{\kappa a\theta_x}{2}\right)}{\frac{\kappa a\theta_x}{2}} = \lim_{\theta_x\to 0}\frac{\cos\left(\frac{\kappa a\theta_x}{2}\right)}{\frac{\kappa a}{2}} = \frac{2}{\kappa a}$$

by l'Hôpital's rule (and similarly for θ_y), so

$$I_0 = I(0,0) = ab\left(\frac{E_0}{\lambda r}\right)^2\left(\frac{2}{\kappa a}\right)^2\left(\frac{2}{\kappa b}\right)^2 \quad ,$$

and

$$I\left(\theta_x,\theta_y\right)=I_0(ab)^2\left(\frac{\kappa}{2}\right)^4\left(\frac{\sin\left(\frac{\kappa a\theta_x}{2}\right)}{\frac{\kappa a\theta_x}{2}}\right)^2\left(\frac{\sin\left(\frac{\kappa b\theta_y}{2}\right)}{\frac{\kappa b\theta_y}{2}}\right)^2 \quad .$$

Problem 2

If we could change the secondary… *Here are the original design parameters of the Hubble Space Telescope:*

	Primary	*Secondary*
Aperture diameter (mm)	*2400*	*266.88*
Apex radius of curvature (mm)	*11040*	*1358*
Eccentricity	*1.00114859*	*1.223110788*
Plate scale	*3.575 arcsec mm^{-1}*	

They are included also in a RayTrace prescription called HST.RAY, on the diskette with which you have been provided. ***(Note: distances are entered in millimeters in HST.RAY.)***

a. *What is the diameter (FWHM, in millimeters) of the Airy disk at* $\lambda = 0.5\ \mu\text{m}$*, in the HST focal plane?*

Let's use subscript 0 for the primary and 1 for the secondary:

$$\Delta x=\frac{\Delta\theta}{PS}=\frac{1}{PS}\frac{1.2\lambda}{D_1}=0.014\ \text{mm}.$$

b. *What is the secondary's magnification,* m*?*

$$m=\frac{\rho}{\rho-k}=\frac{\frac{r_1}{r_0}}{\frac{r_1}{r_0}-\frac{D_1}{D_0}}=10.418 \quad .$$

c. *As you know, the primary mirror was made with an incorrect eccentricity, 1.006578. Suppose that HST's secondary mirror had been accessible to the Shuttle astronauts. Design a new secondary that would correct the spherical aberration in the primary, and leave all the other dimensions of the telescope (including focal plane location and plate scale) the same: report its eccentricity to seven-place accuracy.*

The family of Cassegrain telescopes, with zero third-order SA, is generated by

$$1-\varepsilon_0^2 = \frac{k^4}{\rho^3}\left[\left(\frac{m+1}{m-1}\right)^2 - \varepsilon_1^2\right] \quad ,$$

which, solved for ε_1,

$$\varepsilon_1 = \sqrt{\left(\frac{m+1}{m-1}\right)^2 - \frac{\rho^3}{k^4}\left(1-\varepsilon_0^2\right)}$$

gives us the new secondary eccentricity in terms of the primary eccentricity and the dimensionless telescope parameters. If we keep m, k and ρ the same as in the original design, the focal distances will be the same as in that design; so the correct secondary for the $\varepsilon_0 = 1.006578$ primary has

$$\varepsilon_1 = 1.276904 \quad ,$$

and aperture diameter and apex curvature radius the same as before (see the table above).

d. *Verify with RayTrace that your new design works. First, find the RMS diameter of an on-axis point source in the original HST design. Then change the primary eccentricity to that actually achieved, save the modified prescription in a file called HSTFLAW.RAY on your diskette, and report the RMS diameter of an on-axis point source. Then change the secondary's properties to your new design, save the new prescription in a file called HSTCORR.RAY on your diskette, and report the RMS you get. Compare all the results to the size of the Airy disk from part a.* ***Turn in your diskette with your exam.***

Use 1000 rays and a bullseye pattern in your traces. Do not refocus at any time. Enter your results here. ***(Note: distances are entered in millimeters in HST.RAY.)***

RMS in original design (mm): 0.000036

RMS with flawed primary (mm): 0.357657

RMS with corrected secondary (mm): 0.005560

Airy disk FWHM (mm): 0.014

Does your correction work? <u>Of course it does.</u>

It could do with a little refocusing, as you can see in your values of Sm_RMS, but the blur is already substantially smaller than the Airy disk.

Problem 3

Field curvature in the HST.

a. *Show that the effective focal length of a hyperboloid mirror is $f_{eff} = r/2\varepsilon$, where r is its apex radius of curvature, and ε is its eccentricity.*

For the hyperboloidal mirrors, the focal lengths are given by the usual relationships in terms of *a*, *c*, and the apex radius of curvature *r* and the eccentricity ε:

$$\begin{aligned}
&\varepsilon = \frac{c}{a} \\
&f = c \pm a = a(\varepsilon \pm 1) \\
&r = \frac{b^2}{a} = \frac{c^2 - a^2}{a} = a\left(\varepsilon^2 - 1\right) = a(\varepsilon \pm 1)(\varepsilon \mp 1) = f(\varepsilon \mp 1) \quad ,
\end{aligned}$$

whence

$$\begin{aligned}
&f = \frac{r}{\varepsilon \mp 1} \quad , \\
&\frac{1}{f_{\text{eff}}} = \frac{\varepsilon + 1}{r} + \frac{\varepsilon - 1}{r} = \frac{2\varepsilon}{r} \quad , \text{ and} \\
&f_{\text{eff}} = \frac{r}{2\varepsilon} \quad .
\end{aligned}$$

b. *Calculate the Petzval curvature for the primary and secondary mirrors of the HST, as designed (for which you will need the parameters in the table in Problem 2). From this, calculate the focal length of a field lens made out of flint glass $(n = 1.5)$ that will remove the field curvature if placed immediately before the detector.*

$$\begin{aligned}
&\kappa_P = \frac{2\varepsilon_0}{r_0} + \frac{2\varepsilon_1}{r_1} = 0.01983 \text{ cm}^{-1} \quad , \\
&f_{\text{corr}} = -\frac{1}{n\kappa_P} = -33.62 \text{ cm} \quad .
\end{aligned}$$

That is, it's a diverging lens.

c. *The focal plane will be on the back side of this lens. Keep its back side planar, and use the lensmaker's equation to determine the radius of curvature of the front side.*

$$\frac{1}{f_{\mathrm{corr}}} = (n-1)\left(\frac{1}{r}\right) \qquad \Rightarrow \qquad r = (n-1)f_{\mathrm{corr}} = -168.1 \text{ mm} \quad .$$

The corrector is a planoconvex lens.

d. *In RayTrace, fetch the HST.RAY prescription and add your field lens to it. (It can go where the focal plane is, as long as you make the next surface FI = 0 and AP = 100.) Use zero following thickness; go to the Trace menu; trace 1000 on-axis rays in a bullseye pattern and refocus. Then set DX = 0.01, DY = 0.02, NP = 9, make spot diagrams, and adjust the plate scale so that all the spots fit in the panels.* ***Save this prescription on your diskette as HST_FC.RAY, and hand in the diskette with your exam. (Note: distances are entered in millimeters in HST.RAY.)***

Use the appearance of the spot diagrams to argue that the (original) HST lacks two of the third-order aberrations (besides field curvature, which you just corrected). Which two?

The spot diagram is shown in Figure 1. I used RA = -168.1 mm, EC = 0, FI = 1.5, and FT = 0 for the planoconcave lens surface. Note that the off-axis spots are oblong and centered on the chief ray, and that the blur is very small for on–axis rays. The former fact means there's **little coma** and the latter that there's **little spherical aberration**, and the centered, oblong blur matches the description of astigmatism. That's what Ritchey-Chretien telescopes are *supposed* to be like. Since we've just corrected field curvature, and distortion doesn't blur, the only thing left for the remaining blur to be caused by is astigmatism. (Note to careful readers: our field curvature correction isn't perfect, *because* of astigmatism; it takes a somewhat weaker lens, with RA = -210 mm, to move the focal plane between the tangential and sagittal foci.)

Problem 4

Another way to look at blackbody intensity fluctuations. *Blackbody radiation is such that the components of the electric field are Gaussian random variables with zero average value, which means that the probability for finding the real and imaginary parts of one polarization component E,* $E^{(r)}$ and $E^{(i)}$, *lying in the ranges* $E^{(r)}$ to $E^{(r)} + dE^{(r)}$ *and* $E^{(i)}$ to $E^{(i)} + dE^{(i)}$ *respectively, is*

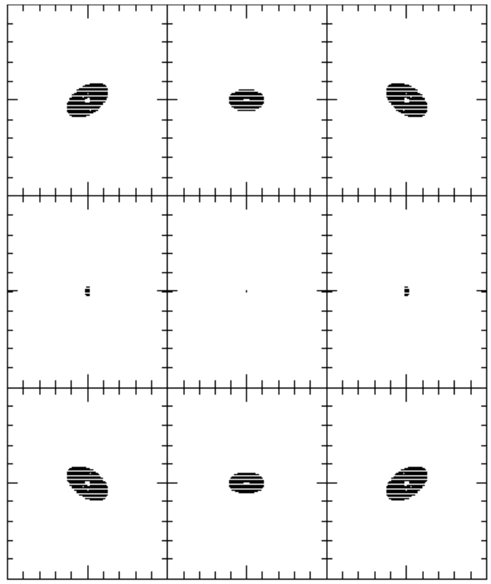

Figure 1: raytrace of the HST with the field-curvature corrector described in part d of problem 3. The angles for the off-axis fields are ±0.01 degrees in x and ±0.02 degrees in y, and the scale is 0.002 mm per tic. Note the absence of spherical aberration and coma, and the presence of (sagittal) astigmatism.

$$p\left(E^{(r)},E^{(i)}\right)dE^{(r)}dE^{(i)} = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{E^{(r)2}+E^{(i)2}}{2\sigma^2}\right]dE^{(r)}dE^{(i)} \quad .$$

a. *Transform this probability distribution to the "polar" coordinates* $E=|E|$ and $\theta = \tan^{-1}\left(E^{(i)}/E^{(r)}\right)$*, and integrate over all* θ *to find the probability* $p(E)dE$ *that the modulus of the electric field lies between E and E+dE.*

Because $dE^{(r)}dE^{(i)} = EdEd\theta$,

$$p(E,\theta)dEd\theta = p\left(E^{(r)},E^{(i)}\right)dE^{(r)}dE^{(i)} = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{E^2}{2\sigma^2}\right]EdEd\theta \quad ;$$

$$p(E)dE = \int_0^{2\pi}\frac{E}{2\pi\sigma^2}\exp\left[-\frac{E^2}{2\sigma^2}\right]dEd\theta = \frac{E}{\sigma^2}\exp\left[-\frac{E^2}{2\sigma^2}\right]dE \quad .$$

b. *From this, show that the probability that the intensity* $I = cE^2/8\pi$ *lies between I and I + dI is*

$$p(I)dI = \frac{4\pi}{c\sigma^2}\exp\left(-\frac{4\pi}{c\sigma^2}I\right)dI \quad .$$

OK, here goes:

$$p(I)dI = p(E)dE = \frac{E}{2\pi\sigma^2}\exp\left[-\frac{E^2}{2\sigma^2}\right]dE \quad ;$$

$$dI = \frac{c}{4\pi}EdE \quad , \text{ so}$$

$$p(I)dI = \frac{4\pi}{c\sigma^2}\exp\left[-\frac{4\pi I}{c\sigma^2}\right]dI \quad .$$

c. *From the probability distribution obtained in part b, show that the standard deviation of the intensity is equal to the average intensity:*

$$\Delta I_{rms} = \sqrt{\overline{I^2} - \bar{I}^2} = \bar{I} \quad .$$

First the mean intensity:

$$\bar{I} = \int_0^\infty Ip(I)dI = \int_0^\infty I\frac{4\pi}{c\sigma^2}\exp\left[-\frac{4\pi I}{c\sigma^2}\right]dI = \frac{c\sigma^2}{4\pi}\int_0^\infty ue^{-u}du$$

$$= \frac{c\sigma^2}{4\pi}\left[-ue^{-u}\Big|_0^\infty + \int_0^\infty e^{-u}du\right] = \frac{c\sigma^2}{4\pi}\left[0 - e^{-u}\Big|_0^\infty\right] = \frac{c\sigma^2}{4\pi} \quad .$$

Then the mean square intensity:

$$\overline{I^2} = \int_0^\infty I^2p(I)dI = \int_0^\infty I^2\frac{4\pi}{c\sigma^2}\exp\left[-\frac{4\pi I}{c\sigma^2}\right]dI = \left(\frac{c\sigma^2}{4\pi}\right)^2\int_0^\infty u^2e^{-u}du$$

$$= \left(\frac{c\sigma^2}{4\pi}\right)^2\left[-u^2e^{-u}\Big|_0^\infty + 2\int_0^\infty ue^{-u}du\right] = \left(\frac{c\sigma^2}{4\pi}\right)^2[0+2] = 2\left(\frac{c\sigma^2}{4\pi}\right)^2$$

$$= 2\bar{I} \quad .$$

Thus

$$\Delta I_{rms} = \sqrt{\overline{I^2} - \bar{I}^2} = \sqrt{2\bar{I}^2 - \bar{I}^2} = \bar{I} \quad .$$

(Q.E.D.)

Problem 5

A 1 m diameter Cassegrain telescope is used for diffraction-limited, spectroscopic observations at a wavelength of 100 μm. The telescope's temperature is 300 K, and its effective emissivity is 0.1. The spectrometer is cold, and has Δλ/λ = 10^{-3} and transmission 0.2. Its detector's quantum efficiency is 0.5, and its gain dispersion is 1.

a. *A noise equivalent power of* 10^{-15} W Hz$^{-1/2}$ *is measured for the telescope-spectrometer combination. How much time, in seconds, is required to reach a signal-to-noise ratio of 10 on an object that emits a narrow spectral line with flux (power per unit telescope area)* 10^{-18} W cm^{-2}*?*

$$\frac{S}{N} = \frac{P_S}{NEP\sqrt{\Delta f}} = \frac{P_S}{NEP}\sqrt{2\Delta t}$$

$$\Delta t = \frac{1}{2}\left[\left(\frac{S}{N}\right)\frac{NEP}{P_S}\right]^2 = \frac{1}{2}\left[(10)\frac{10^{-15}\ \mathrm{W\ Hz^{-1/2}}}{10^{-18}\ \mathrm{W\ cm^{-2}} \times \pi\left(\frac{100\ \mathrm{cm}}{2}\right)^2}\right]^2 = 0.8\ \mathrm{sec}\quad .$$

b. *Is the spectrometer background limited? Prove your assertion with a calculation.*

The detector is the same size as the FWHM diffraction spot, so $A\Omega = \lambda^2$, and $\Delta\lambda / \lambda = \Delta\nu / \nu = 1/1000$, so

$$P_B = \varepsilon B_\nu(T)\Delta\nu A\Omega = 2\varepsilon h\nu\overline{N}\frac{\nu}{1000}\quad ,$$

where

$$\overline{N} = \frac{1}{e^{h\nu/kT} - 1} = 1.63$$

for T = 300 K and $\lambda = 100\ \mu\mathrm{m}$. Thus the background-limited NEP is

$$NEP = \frac{1}{1-\varepsilon}\sqrt{\frac{\beta}{\tau\eta}2h\nu\cdot 2\varepsilon h\nu\overline{N}\frac{\nu}{100}\left(1+\frac{\tau\eta\varepsilon}{\beta}\overline{N}\right)} = 3.1\times 10^{-16}\ \mathrm{W\ Hz^{-1/2}}\quad .$$

The spectrometer is not background limited. For shame! Those who built it should have worked harder to eliminate non-fundamental sources of noise.

Problem 6

A photodetector used at wavelength λ is made out of material with refractive index n_0 and used in vacuum.

a. *What is its maximum quantum efficiency (that is, what is the transmission of its front surface at normal incidence)?*

We did this in class as an example (lecture 26, 12/2/99). The starting and finishing media have

$$Y_0 = 1 \quad Y_2 = \sqrt{\frac{\varepsilon}{\mu}} = n_0 \quad ,$$

and they are separated by a medium of zero thickness:

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad .$$

Thus two of the terms in the denominator of t vanish:

$$t = \frac{2Y_0}{Y_0 + Y_2} = \frac{2}{1+n_0} \quad ;$$

$$\tau = \frac{Y_2}{Y_0}|t|^2 = n_0\left(\frac{2}{1+n_0}\right)^2 \leq 1 \quad ,$$

with the = holding only if the detector is made of vacuum, which doesn't sound very useful.

b. *Prove that the maximum quantum efficiency can be made 100% for normal incidence at wavelength λ by coating the detector surface with a dielectric film of index $n_1 = \sqrt{n_0}$ and thickness $d = \lambda/4n_1$.*

Now medium 1 has finite thickness, but magically with the given choice of d the other pair of characteristic-matrix elements vanishes:

$$M_1 = \begin{bmatrix} \cos\kappa\ell & -i\sin\kappa\ell / Y \\ -iY\sin\kappa\ell & \cos\kappa\ell \end{bmatrix} = \begin{bmatrix} \cos\frac{2\pi nd}{\lambda} & -\frac{i}{n}\sin\frac{2\pi nd}{\lambda} \\ -in\sin\frac{2\pi nd}{\lambda} & \cos\frac{2\pi nd}{\lambda} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\frac{\pi}{2} & -\frac{i}{\sqrt{n_0}}\sin\frac{\pi}{2} \\ -i\sqrt{n_0}\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{i}{\sqrt{n_0}} \\ -i\sqrt{n_0} & 0 \end{bmatrix} \quad .$$

Thus

$$t = \frac{2}{\frac{-i}{\sqrt{n_0}}\, n_0 - i\sqrt{n_0}} = -\frac{1}{i\sqrt{n_0}} \quad ,$$

$$\tau = n_0 |t|^2 = n_0 \frac{1}{n_0} = 1 \quad .$$

(Q.E.D.)