Astronomy 203 Problem Set #1: Solutions

16 September 1999

1. *Consider the situations, shown in the figure below, of dielectric media with refractive indices* n_1 *and* $n_2 > n_1$. *For the convex surface, show that*

$$\frac{n_1}{o} + \frac{n_2}{i} = \frac{n_2 - n_1}{r} \quad ,$$

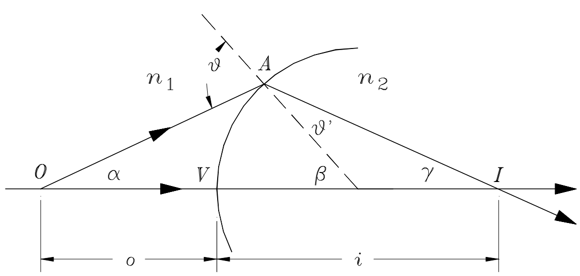
and for the concave surface, show that

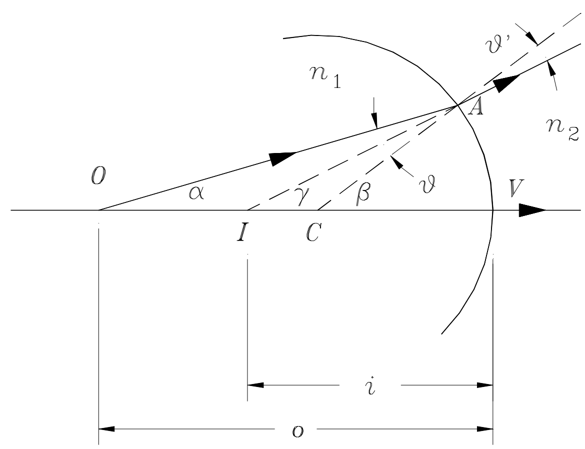
$$\frac{n_1}{o} + \frac{n_2}{i} = -\frac{n_2 - n_1}{r} \quad ,$$

and hence that the focal length is

$$f = \frac{n_2}{n_2 - n_1} r \quad .$$

subject to the sign convention $r > 0$ *for the convex surface and* $r < 0$ *for the concave one.*





These demonstrations are so similar that it will be convenient to do them both at once, and simply to display equations pertaining to the left-hand figure on the left, and those pertaining to the right-hand figure on the right. In both figures, the ray aimed at the center and vertex is incident normally on the sphere ($\theta_i = 0$, or $\sin\theta_i = 0$), and it passes through without deviation ($\sin\theta_i = 0$). The other ray, incident on the sphere at angle θ, refracts such that

$$n_1 \sin\theta = n_2 \sin\theta' \quad .$$

We intend to use the paraxial approximation, in which all of the angles labelled in both figures are small. For a small angle, $x << 1$, we know that $\sin x \cong x$ to good approximation, so

$$\theta' = \frac{n_1}{n_2}\theta \quad ,$$

so we can write the following expressions relating the angles in the figure:

$$\alpha + \beta = \theta \qquad\qquad \alpha + \theta = \beta$$

$$\gamma + \theta' = \gamma + \frac{n_1}{n_2}\theta = \beta \qquad\qquad \gamma + \theta' = \gamma + \frac{n_1}{n_2}\theta = \beta$$

Eliminating θ, we find

$$\gamma + \frac{n_1}{n_2}(\alpha + \beta) = \beta \qquad\qquad \gamma + \frac{n_1}{n_2}(\beta - \alpha) = \beta$$

$$n_2\gamma + n_1\alpha = (n_2 - n_1)\beta \qquad\qquad n_1\alpha - n_2\gamma = -(n_2 - n_1)\beta$$

Once again we can express these angles in terms of their radian measure, using the arclength AV:

$$\alpha \approx \frac{AV}{o} \quad \beta \equiv \frac{AV}{r} \quad \text{and} \quad \gamma \approx \frac{AV}{i} \quad .$$

and the common factor of the length of AV cancels out:

$$\frac{n_1}{o} + \frac{n_2}{i} = \frac{n_2 - n_1}{r} \qquad\qquad \frac{n_1}{o} - \frac{n_2}{i} = -\frac{n_2 - n_1}{r}$$

from which we can define a focal length $f = \lim\limits_{o \to \infty} i$ that comes out to

$$f = \frac{n_2}{n_2 - n_1} r \, .$$

The two expressions abouve that relate o, i, and r can be unified by establishing the sign convention $r > 0$ for convex surfaces, $r < 0$ for concave surfaces, $i > 0$ on the right side of the surface and $i < 0$ on the left side, in both figures.

2. a. *Derive an expression for the back focal length of the following combination: a thin lens with focal length* f_1 *and a convex spherical dielectric of radius r and index n, placed a distance d apart and illuminated from the lens' side.*

   Assume I meant that the two optical elements are in vacuum.

   Let's begin by considering the an object lying at very large distance; then the back focal length f is equal to the final image distance i. Then the image formed by the first lens lies at the lens' focal point. The distance from this image to the apex of the dielectric is $d - f_1$ (>0) if it lies between the optical elements – that is, if it is a real object from the dielectric sphere's point of view. The distance to the apex is $f_1 - d$ (>0) if the image lies on the far side of the surface, but in this case it is a virtual object to the dielectric sphere, for which we would assign a negative algebraic sign to the object distance. Either way, $o = d - f_1$. From the results of Problem 1 (the left-hand one, that is), we get

$$\frac{1}{d-f_1}+\frac{n}{f}=\frac{n-1}{r} \quad \text{, or}$$

$$f=\frac{nr(d-f_1)}{(n-1)(d-f_1)-r} \quad .$$

Note that this isn't quite the same as the result for two thin lenses in vacuum.

b. *I am nearsighted, and am always seen wearing either contact lenses or eyeglasses. The correct power for my glasses is determined by my optometrist to be –2.25 diopters, by placing a variety of lenses in the normal "eyeglasses" position, 1 cm in front of each eye. What is the correct power for my contact lenses?*

The back focal distance f is the same when I wear my eyeglasses as when I wear my contact lenses, because my eyes have the same shape and size (in particular, they're in their "relaxed" configuration) in the two conditions. Thus, from the results of part a, we get

$$\frac{1}{f}=\frac{n-1}{nr}-\frac{1}{n\left(d-f_{\text{glasses}}\right)}=\frac{n-1}{nr}+\frac{1}{nf_{\text{contacts}}} \quad \text{, or}$$

$$f_{\text{contacts}}=f_{\text{glasses}}-d \quad .$$

We are told that $f_{\text{glasses}}=(-1/2.25)\text{ m} = 0.44\text{ m}$ and that d = 1 cm, so and

$$f_{\text{contacts}}=-0.45\text{ m},$$
$$\frac{1}{f_{\text{contacts}}}=-2.20\text{ diopters.}$$

If you've ever had an eye exam, you know that not very many people can reproducibly detect a 0.05 diopter change in power; that is, the difference between the proper contact-lens prescription and the eyeglasses prescriptions is well within the experimental uncertainties. Thus most people are given the same prescription for glasses and contacts. (In fact, Bausch and Lomb's standard contact lenses only come in 0.25 diopter intervals.)

c. *The position of the most distant object on which I can focus, when I'm not wearing my glasses or contacts, is called my far point. How far away from my eyes is my far point?*

My eyes are the same shape and size while looking at their far point, *unaided*, as they are looking at infinity through glasses or contacts; thus the final image distance is the same in the two cases. Without correction this is determined by

$$\frac{1}{o}+\frac{n}{i}=\frac{n-1}{r} \quad ,$$

and since this i is equal to the f determined previously, we can use one of the results of part b to get the object distance (to the far point):

$$\frac{1}{no}+\frac{1}{f}=\frac{1}{no}+\frac{n-1}{nr}-\frac{1}{nf_{\text{contacts}}}=\frac{n-1}{nr} \quad \text{, or}$$

$$o = -f_{\text{contacts}} = 0.45\text{ m.} \quad .$$

3. *Derive an expression for the back focal length of a thick biconvex lens: a lens for which the paraxial approximation applies to the surfaces, but for which the distance d between the apices is finite.*

   Let's take the lens to have index *n*, surfaces with radius of curvature *r*, and to be used in vacuum. The front surface is convex, so its radius of curvature enters as a positive number. The back surface looks concave to the incident light, and its radius of curvature therefore needs to be put in as a negative number. To make this clear we'll consider the term *r* to be positive, and the curvature radii of the surfaces to be +*r* and –*r*. Since we're only interested in the focal length, we can let *o* approach infinity right away. With all this we can write, for the position of the image formed by the first surface in the absence of the second,

$$\lim_{o\to\infty}\left(\frac{1}{o}+\frac{n}{i}\right)=\frac{n}{i}=\frac{n-1}{r} \quad , \text{ or}$$

$$i=\frac{nr}{(n-1)} \quad .$$

   As usual, depending upon the distance *d* between the apices, the first image could lie on either side of the second surface. If the image happens to lie in between the lenses, then the next object distance is simply $o' = d - i$. If on the other hand the image lies to the right of the second surface it comprises a virtual object for the second surface, for which the object distance needs to be written as $o' = -(i-d)$. No matter what, the second object distance amounts to $d - i$, so the second image distance is given by the solution to

$$\frac{n}{d-i}+\frac{1}{i'}=\frac{1-n}{-r} \quad .$$

   This second image distance is none other than the back focal distance, since we earlier placed the object at infinity; it comes out to

$$\frac{1}{f}=\frac{1}{i'}=\frac{n-1}{r}-\frac{n}{d-i}=\frac{(n-1)(d-i)-nr}{r(d-i)}=\frac{(n-1)\left(d-\frac{nr}{n-1}\right)-nr}{r\left(d-\frac{nr}{n-1}\right)}$$

$$=\frac{n-1}{r}\frac{d(n-1)-2nr}{d(n-1)-nr}$$

   (note at this point that for *d* = 0 the result expected from the lensmakers' equation, $1/f=(n-1)(2/r)$, is obtained), or

$$f=\frac{r}{n-1}\frac{d(n-1)-nr}{d(n-1)-2nr} \quad .$$

4. Hecht problem 5.33: *Two thin lenses having focal lengths of +15.0 cm and –15.0 cm are positioned 60.0 cm apart. A page of print is held 25.0 cm in cront of the positive lens. Describe, in detail, the image of the print (i.e., insofar as it's paraxial).*

We could be more detailed if we knew the words that are printed on this page; with what we're told, all we can get is the position and magnification of the final image.

$$o_1 = 25 \text{ cm},\ f_1 = 15 \text{ cm} \qquad o_2 = d - i_1 = 22.5 \text{ cm},\ f_2 = -15 \text{ cm}$$

$$i_1 = \frac{o_1 f_1}{o_1 - f_1} = 37.5 \text{ cm} \qquad i_2 = \frac{o_2 f_2}{o_2 - f_2} = -9 \text{ cm}$$

$$m_1 = -\frac{i_1}{o_1} = -1.5 \quad , \qquad m_2 = -\frac{i_2}{o_2} = -0.4 \quad ,$$

$$m = m_1 m_2 = -0.6 \quad .$$

The page appears to an observer on the far side of the lenses to lie 9 cm behind the second lens (that is, between the lenses), and the print looks upside down and 60% of its original size.

5 *The 200-inch (5 m) telescope at Palomar Observatory has a primary mirror with focal length 16.7 m. It is most often used with a Cassegrain focus behind the primary, a distance 3 m from the primary's apex.*

a. *The secondary mirror's apex is 89 cm from the prime focus. What are its apex curvature and eccentricity?*

A Cassegrain telescope has a paraboloid primary and a convex hyperboloid secondary, with the focus of the primary coincident with the near focus of the secondary. The sum of the secondary focal lengths equals the distance between the prime and Cassegrain foci. Thus the shorter focal length of the secondary is 0.89 m, and the longer one is (16.7 m + 3 m) – 0.89 m = 18.81 m. Now note, from Equation 3.13, that

$$f_1 = c + a = 18.81 \text{ m} \qquad f_2 = c - a = 0.89 \text{ m}$$

Adding these two expressions we get

$$c = \frac{f_1 + f_2}{2} = 9.85 \text{ m} \quad ,$$

subtracting them we get

$$a = \frac{f_1 - f_2}{2} = 8.96 \text{ m} \quad ,$$

and from the values of c and a we finally get the apex curvature and eccentricity:

$$\kappa = \frac{a}{b^2} = \frac{a}{c^2 - a^2} = 0.535 \text{ m}^{-1} \qquad (r = 1.868 \text{ m}),$$

$$\varepsilon = \frac{c}{a} = 1.099 \quad .$$

b. *What is plate scale at the Cassegrain focus?*

Consider a celestial object of small angular extent $\Delta\theta$. In terms of the primary's plate scale,

$$PS = \frac{1}{f_{\text{primary}}} = \frac{1}{16.7} \text{ radians m}^{-1} = 6.18 \text{ arcsec mm}^{-1} \quad ,$$

the length of the image in the primary's focal plane is $\Delta y = \Delta\theta/PS = f_{\text{primary}}\Delta\theta$. This image is re-imaged by the secondary; the relevant object and image distances are simply the secondary focal lengths $f_2 = 0.89\ \text{m}$ and $f_1 = 18.81\ \text{m}$, respectively. Thus the magnification from prime to Cassegrain focus is $m = -i/o = -f_1/f_2$ (remember the hyperboloid sign convention!), and the length of the final image is

$$\Delta y' = m\Delta y = -\frac{f_1}{f_2} f_{\text{primary}}\Delta\theta \quad .$$

Thus the Cassegrain plate scale is

$$PS' = \frac{\Delta\theta}{\Delta y'} = -\frac{f_2}{f_1 f_{\text{primary}}} = -2.83\times10^{-3}\ \text{radians m}^{-1} = -0.29\ \text{arcsec mm}^{-1} \quad .$$

The minus sign indicates that images in the Cassegrain focal plane are inverted with respect to those at prime focus.

c. *Estimate the diameter d of the secondary mirror, and the final focal ratio* $F = f/d$, *where* f *is the relevant focal length of the secondary.*

A rough estimate is provided by inspection of Figure 3.8 in the lecture notes. The mirror diameters and focal lengths are in the ratio

$$\frac{d}{f_2} = \frac{d_{\text{primary}}}{f_{\text{primary}}} \quad \Rightarrow \quad d = 27\ \text{cm}$$

if on-axis rays incident at the edge of the primary are to be reflected by the secondary. The final focal ratio comes out to $F = f_1/d = 70$.

The secondary would need to be a little bit larger than this if it were really to intercept light incident at the edge of the primary, at nonzero angle with respect to the optical axis; the precise value of the diameter depends upon the angular diameter of the field of view desired by the designer. (This turns out not to be the way the $F/70$ secondary at Palomar is designed, though; it's a bit *smaller* in diameter than one would expect, for reasons having to do with the performance of the telescope at infrared wavelengths. We will discuss this later on, when we come to talk about optimization of telescopes and optics for specific wavelength ranges.)