Astronomy 203 Problem Set #5: Solutions

2 November 1999

1. *Using RayTrace 5.0 and the prescriptions developed in Problem Set #3, problem 3, generate spot diagrams, on the same scale, for on-axis rays in the focal plane of the Hubble Space Telescope, as designed and as originally realized. (Or use the plots you generated in that problem set…) On each of these plots, draw circles to indicate the FWHM diameter of the central diffraction maximum, and the diameter of the first dark ring, for a wavelength of λ = 0.55 μm. Comment on the effect the errors in the primary mirror must have had on early observations with HST.*

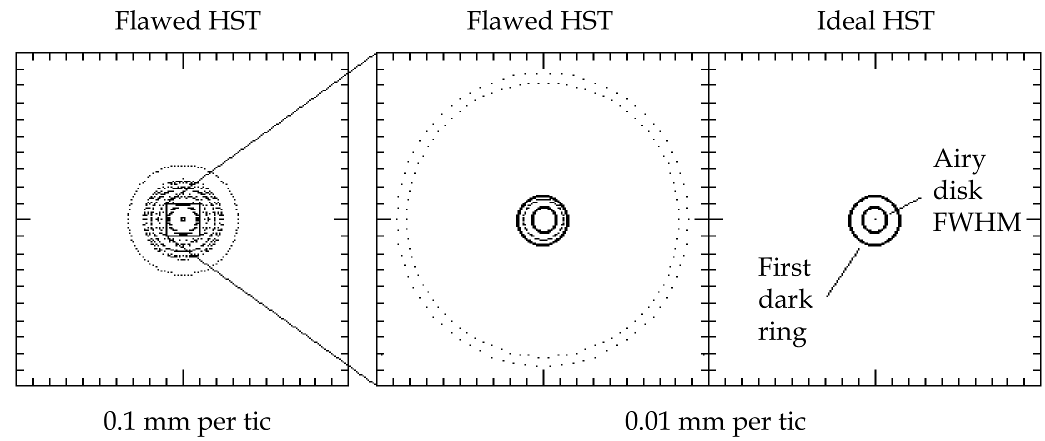

Figure 1: Comparison of ray traces for the original, "real" and the corrected or "ideal" Hubble Space Telescope (points) with the effects of diffraction (bold circles in the center and right-hand panel). Details of the prescription are given in the solutions to Problem Set #3.

From Figure 13.4 in the lecture notes we see that the FWHM diameter of the central maximum of the Airy pattern is about the same as the *radius* of the first dark ring; at this wavelength, that's $1.2\lambda/D$ = 2.75×10^{-7} radians $=5.7\times10^{-2}$ arcseconds, which, with a plate scale of 3.575 arcseconds per millimeter, comes to about 0.016 mm in the focal plane. Circles with this diameter and this radius are overlaid on pre-repair and ideal HST spot diagrams in Figure 1. Note that the blur from diffraction is much greater than that from aberrations for the ideal HST, but is dwarfed by the aberrations in the pre-repair HST (RMS diameter 0.37 arcseconds – see the solutions to problem set 3). The telescope was supposed to be diffraction-limited, but instead had turned out to be SA-limited, and the resolution is degraded by a large factor. It was still better than the usual ground-based telescopes, but nowhere near as good as it was supposed to be. For instance: it was hoped that HST could be used to resolve individual Cepheid variable stars in galaxies 20 Mpc away, thus to eliminate a few of the weaker rungs in the extragalactic distance-scale "ladder". With its range reduced to about 2 Mpc by the flaw, this, among other important projects, could not be attempted until the repair mission had succeeded. Now Cepheids have been detected by HST in M101 (about 20 Mpc away, and a member of the Virgo cluster), among many other galaxies similarly distant, and we have a much more reliable Hubble constant than we did before.

2. Gaussian beams stay Gaussian as they propagate. *Show that a Gaussian near-field distribution with 1/e radius ρ_N,*

$$E_N(x',y') = E_0 e^{-i\omega t} \exp\left(-\frac{x'^2+y'^2}{\rho_N^2}\right) \quad ,$$

gives rise to a Gaussian far-field distribution,

$$E_F(x,y,z) = \frac{\pi\rho_N^2 E_0}{\lambda z} e^{i(\kappa z-\omega t)} \exp\left(-\frac{\pi^2\rho_N^2\left(x^2+y^2\right)}{\lambda^2 z^2}\right) \quad .$$

that has 1/e radius $\lambda z/\pi\rho_N$.

The diffraction integral is simply

$$E_F(x,y,z) = \frac{e^{i(\kappa z-\omega t)}}{\lambda z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_0 \exp\left[-\frac{x'^2+y'^2}{\rho_N^2}\right]\exp\left[-i\left(\kappa_x x'+\kappa_y y'\right)\right]dx'dy' \tag{1}$$

Completing the square in the exponent, we have, for x',

$$\begin{aligned} -\frac{x'^2}{\rho_N^2} - i\kappa_x x' &= -\left(\frac{x'^2}{\rho_N^2} + i\kappa_x x' + \left(\frac{i\kappa_x\rho_N}{2}\right)^2\right) + \left(\frac{i\kappa_x\rho_N}{2}\right)^2 \\ &= -\left(\frac{x'}{\rho_N} + \frac{i\kappa_x\rho_N}{2}\right)^2 - \frac{\kappa_x^2\rho_N^2}{4} \quad , \end{aligned} \tag{2}$$

and similarly for y', so that the diffraction integral becomes

$$\begin{aligned} E_F(x,y,z) &= \frac{E_0 e^{i(\kappa z-\omega t)}}{\lambda z}\exp\left(-\frac{\left(\kappa_x^2+\kappa_y^2\right)\rho_N^2}{4}\right) \\ &\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\left(\frac{x'}{\rho_N}+\frac{i\kappa_x\rho_N}{2}\right)^2\right)\exp\left(-\left(\frac{y'}{\rho_N}+\frac{i\kappa_y\rho_N}{2}\right)^2\right)dx'dy' \end{aligned} \tag{3}$$

Now change variables:

$$\begin{aligned} u &= \frac{x'}{\rho_N} + \frac{i\kappa_x\rho_N}{2} \qquad du = \frac{dx'}{\rho_N} \qquad -\infty < u < \infty \\ v &= \frac{y'}{\rho_N} + \frac{i\kappa_y\rho_N}{2} \qquad dv = \frac{dy'}{\rho_N} \qquad -\infty < v < \infty \end{aligned} \quad , \tag{4}$$

and we have

$$E_F(x,y,z) = \frac{\rho_N^2 E_0 e^{i(\kappa z-\omega t)}}{\lambda z}\exp\left(-\frac{\left(\kappa_x^2+\kappa_y^2\right)\rho_N^2}{4}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(u^2+v^2\right)}du\,dv \quad . \tag{5}$$

Change these variables, in turn, to the equivalent polar coordinates:

$$r^2 = u^2 + v^2 \qquad \tan\varphi = \frac{v}{u} \qquad du\,dv = r\,dr\,d\varphi \qquad 0 < r < \infty \qquad 0 < \varphi < 2\pi \tag{6}$$

to make the diffraction integral look like

$$E_F(x,y,z) = \frac{\rho_N^2 E_0 e^{i(\kappa z-\omega t)}}{\lambda z}\exp\left(-\frac{\left(\kappa_x^2+\kappa_y^2\right)\rho_N^2}{4}\right)\int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} re^{-r^2}dr \quad . \tag{7}$$

The integral over φ simply comes out to 2π. Now make one more variable change in the other integral:

$$w = r^2 \qquad dw = 2r\,dr \qquad 0 < w < \infty \quad , \tag{8}$$

and it becomes trivial:

$$\begin{aligned} E_F(x,y,z) &= \frac{\pi\rho_N^2 E_0 e^{i(\kappa z-\omega t)}}{\lambda z}\exp\left(-\frac{\left(\kappa_x^2+\kappa_y^2\right)\rho_N^2}{4}\right)\int_0^{\infty} e^{-w}dw \\ &= \frac{\pi\rho_N^2 E_0 e^{i(\kappa z-\omega t)}}{\lambda z}\exp\left(-\frac{\left(\kappa_x^2+\kappa_y^2\right)\rho_N^2}{4}\right) \end{aligned} \quad . \tag{9}$$

But $\kappa_x = \kappa x / z$ and $\kappa_y = \kappa y / z$, so this is just the desired result:

$$E_F(x,y,z) = \frac{\pi\rho_N^2 E_0}{\lambda z} e^{i(\kappa z-\omega t)} \exp\left(-\frac{\pi^2\rho_N^2\left(x^2+y^2\right)}{\lambda^2 z^2}\right) \quad , \tag{10}$$

a Gaussian, with $1/e$ width $\lambda z/\pi\rho_N$.

3. *Most telescope primary mirrors have central obscurations, in addition to being circular, so their diffraction patterns differ somewhat from Equations 13.14-15.*

   a. *Derive an expression the far-field intensity as a function of $\kappa a\theta$ for an annular aperture, with outer half-diameter a and inner half diameter ka (k < 1).*

   The annulus is still axisymmetric, so the calculation goes just the same as that for the filled circular aperture in Lecture 13, except for the bounds on the radial integral. We can therefore join the derivation at the point of that integration, Equation 13.6, and change the lower integration bound from $r' = 0$ to $r' = ka$, or

$$E_F(q,t) = \frac{2\pi E_{N0} e^{i(\kappa r-\omega t)}}{\lambda r}\int_0^a dr' r' J_0\left(\frac{\kappa q r'}{r}\right) = \frac{2\pi E_{N0} e^{i(\kappa r-\omega t)}}{\lambda r}\left(\frac{r}{\kappa q}\right)^2 \int_{\kappa k a q/r}^{\kappa a q/r} v J_0(v)dv \quad . \tag{11}$$

   Again we integrate by use of Equation 13.7, the recurrence relation between Bessel functions of different order:

$$\frac{d}{du}\left[u^m J_m(u)\right] = u^m J_{m-1}(u) \quad , \tag{12}$$

and again choose m = 1, but this time integrate it over different bounds:

$$uJ_1(u) - u'J_1(u') = \int_{u'}^{u} vJ_0(v)dv \quad . \tag{13}$$

We substitute this result into Equation (11) and obtain

$$\begin{aligned} E_F(q,t) &= \frac{2\pi E_{N0} e^{i(\kappa r - \omega t)}}{\lambda r}\left(\frac{r}{\kappa q}\right)^2\left[\left(\frac{\kappa a q}{r}\right)J_1\left(\frac{\kappa a q}{r}\right) - \left(\frac{\kappa k a q}{r}\right)J_1\left(\frac{\kappa k a q}{r}\right)\right] \\ &= \frac{\pi a^2 E_{N0} e^{i(\kappa r - \omega t)}}{\lambda r}\left[\frac{r}{\kappa a q}2J_1\left(\frac{\kappa a q}{r}\right) - k^2\left(\frac{r}{\kappa k a q}\right)2J_1\left(\frac{\kappa k a q}{r}\right)\right] \quad , \end{aligned} \tag{14}$$

or

$$E_F(\kappa a\theta, k, t) = \frac{E_{N0} A e^{i(\kappa r - \omega t)}}{\lambda r}\left[\frac{2J_1(\kappa a\theta)}{\kappa a\theta} - k^2\frac{2J_1(\kappa k a\theta)}{\kappa k a\theta}\right] \quad ,$$

where we have substituted $A = \pi a^2$ and $\theta = q / r$, as before. The intensity on the screen becomes

$$I_F(\kappa a\theta, k) = \frac{c}{8\pi}E_F^*(\kappa a\theta, k, t)E_F(\kappa a\theta, k, t) = \frac{cE_{N0}^2 A^2}{8\pi\lambda^2 r^2}\left[\frac{2J_1(\kappa a\theta)}{\kappa a\theta} - k^2\frac{2J_1(\kappa k a\theta)}{\kappa k a\theta}\right]^2 \quad . \tag{15}$$

Recall, from Equation (13.13), that

$$\lim_{u\to 0}\frac{J_1(u)}{u} = \frac{1}{2} \quad , \tag{16}$$

and we can write, for the intensity at $\theta = 0$,

$$I_F(0,k) = \frac{cE_{N0}^2 A^2}{8\pi\lambda^2 r^2}\left[1 - k^2\right]^2 \quad . \tag{17}$$

Finally, we use Equation (17) in Equation (15) and obtain

$$I_F(\kappa a\theta, k) = I(0,k)\left[\frac{\dfrac{2J_1(\kappa a\theta)}{\kappa a\theta} - k^2\dfrac{2J_1(\kappa k a\theta)}{\kappa k a\theta}}{1 - k^2}\right]^2 \quad . \tag{18}$$

b. *Plot the intensity divided by peak intensity,* $I(\kappa a\theta, k)/I(0)$*, against* $\kappa a\theta$*, on the same plot with* $I(\kappa a\theta)/I(0)$ *for the filled circular aperture, as given by Equations 13.14-15, for k = 0.2 – a rather typical value for telescopes, and used in our Cassegrain telescope example in §9.2 – and a more extreme value, k = 0.9. What are the major differences between the diffraction patterns of filled circular apertures and annular apertures?*

See Figure 2. For small k the difference between the diffraction patterns of the filled aperture and the annulus are pretty small. For larger values of k one sees the Airy disk and the dark and bright rings shrink in diameter, and the intensity of the bright rings increase relative to the peak.

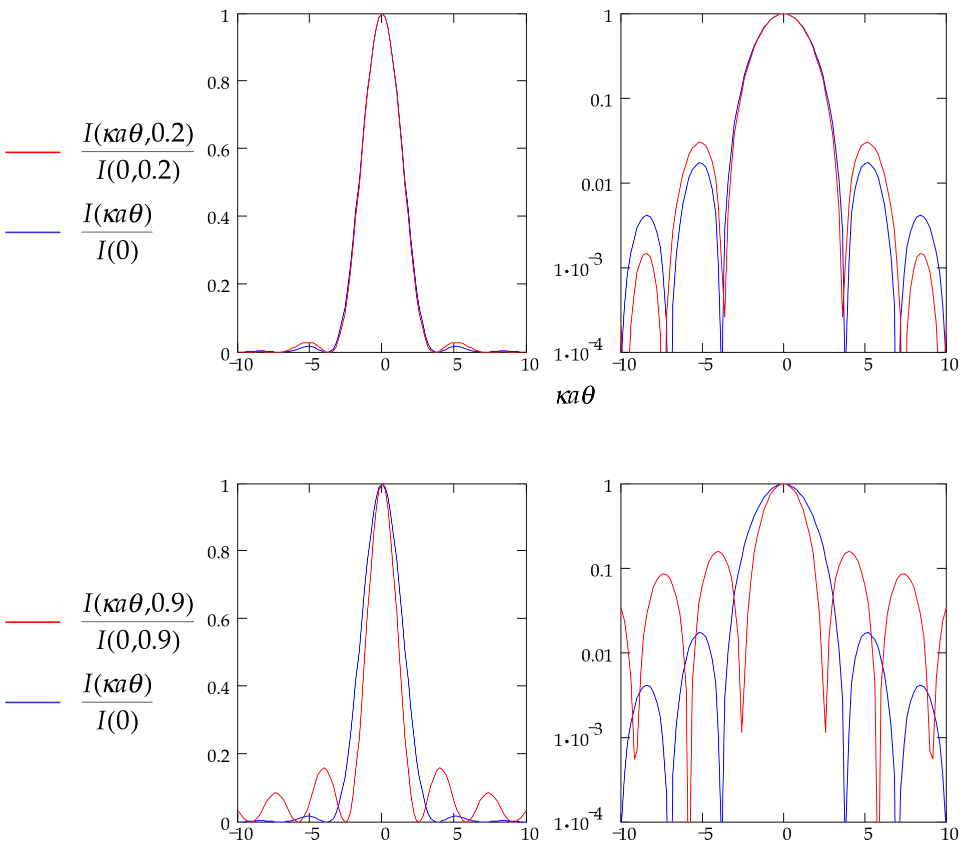

Figure 2: far-field intensity relative to peak for diffraction from a filled circular aperture (blue curves) and from a circular annulus (red curves) for two different values of the ratio of the annulus' inner and outer diameter, k = 0.2 and 0.9.

4. *Show that a pair of thin lenses separated by the sum of their focal lengths, with a waist of the input beam at the focal point of one of the lenses, produces an output beam whose waist location is independent of wavelength, and derive the ratio of the output and input beam waist diameters. (This arrangement is usually called a "Gaussian beam telescope", and is useful in radio-astronomy applications in which a large frequency or wavelength range must be used.)*

Start with a beam waist (radius ρ_{01}) at the focus of a lens with focal length f_1. The image waist lies a distance d_2 on the opposite side of the lens:

$$\frac{d_2}{f_1} = 1 + \frac{\frac{d_1}{f_1} - 1}{\left(\frac{d_1}{f_1} - 1\right)^2 + \left(\frac{\pi \rho_{01}{}^2}{\lambda f_1}\right)^2} = 1 + \frac{\frac{f_1}{f_1} - 1}{\left(\frac{f_1}{f_1} - 1\right)^2 + \left(\frac{\pi \rho_{01}{}^2}{\lambda f_1}\right)^2} = 1 \quad , \tag{19}$$

or $d_2 = f_1$. The image beam waist has a new size, ρ_{02}. The image position is $d_1' = f_2$ away from the next lens, so the final beam waist will thus be a distance d_2' on the other side of the second lens, given by

$$\frac{d_2'}{f_2} = 1 + \frac{\frac{d_1'}{f_2} - 1}{\left(\frac{d_1'}{f_2} - 1\right)^2 + \left(\frac{\pi\rho_{02}^2}{\lambda f_2}\right)^2} = 1 + \frac{\frac{f_2}{f_2} - 1}{\left(\frac{f_2}{f_2} - 1\right)^2 + \left(\frac{\pi\rho_{02}^2}{\lambda f_2}\right)^2} = 1 \quad , \tag{20}$$

or $d_2' = f_2$ – independent of wavelength.

The first image waist radius is given in terms of the original waist radius as

$$\left(\frac{\rho_{02}}{\rho_{01}}\right)^2 = \frac{1}{\left(\frac{d_1}{f_1} - 1\right)^2 + \left(\frac{\pi\rho_{01}^2}{\lambda f_1}\right)^2} = \frac{1}{\left(\frac{f_1}{f_1} - 1\right)^2 + \left(\frac{\pi\rho_{01}^2}{\lambda f_1}\right)^2} = \left(\frac{\lambda f_1}{\pi\rho_{01}^2}\right)^2 \quad , \tag{21}$$

or

$$\rho_{02} = \frac{\lambda f_1}{\pi\rho_{01}} \quad , \tag{22}$$

which is not independent of wavelength. For the final beam waist radius, however, we have

$$\left(\frac{\rho_{03}}{\rho_{02}}\right)^2 = \frac{1}{\left(\frac{d_2}{f_2} - 1\right)^2 + \left(\frac{\pi\rho_{02}^2}{\lambda f_2}\right)^2} = \frac{1}{\left(\frac{f_2}{f_2} - 1\right)^2 + \left(\frac{\pi\rho_{02}^2}{\lambda f_2}\right)^2} = \left(\frac{\lambda f_2}{\pi\rho_{02}^2}\right)^2 \quad , \tag{23}$$

or

$$\rho_{03} = \frac{\lambda f_2}{\pi\rho_{02}} = \frac{\lambda f_2}{\pi\left(\frac{\lambda f_1}{\pi\rho_{01}}\right)} = \frac{f_2}{f_1}\rho_{01} \quad , \tag{24}$$

which *is* wavelength-independent. The magnification of the beam waists is given simply by the ratio of the focal lengths. (Note that this the same result that one gets in geometrical optics.)