# Astronomy 203 Problem Set #6: Solutions

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1. *A cryogenic spectrometer exposes a far-infrared detector that has resistance $5\times10^6$ Ω, quantum efficiency η = 1, and photoconductive gain G = 1, to radiation from a 300 K blackbody, in a diffraction-limited beam and a relative bandwidth of Δν/ν = 1/100 at a wavelength of 100 μm. The detector's temperature is 4.2 K. Calculate the signal current through the detector from the blackbody radiation, the RMS shot noise current per square root bandwidth, and the RMS current per square root bandwidth due to Johnson noise. What is the largest source of noise?*

   Most of the power in a diffraction-limited beam is contained within the Airy disk, for which $A\Omega \cong 4\lambda^2$ (see Equation 15.6), so the power that the blackbody (temperature $T_{\rm BB}$ =300 K) shines on the detector is given by

$$P = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT_{\rm BB}}-1}\Delta\nu\, A\Omega = \frac{8h\nu\Delta\nu}{e^{h\nu/kT_{\rm BB}}-1} \quad . \tag{1}$$

   In turn, photocurrent is generated in the amount

$$I_S = \frac{\eta G q P_S}{h\nu} = \frac{8\eta q\Delta\nu}{e^{h\nu/kT_{BB}}-1} = 6.2\times10^{-8}\ \text{A} \quad , \tag{2}$$

   since the frequency bandwidth of the light falling on the detector is Δν = ν/100 = $c$/100λ. The RMS shot noise photocurrent per square root bandwidth is

$$\frac{\Delta I_{\rm rms}(\text{shot})}{\sqrt{\Delta f}} = \sqrt{2qI_S} = 1.4\times10^{-13}\ \text{A Hz}^{-1/2} \quad . \tag{3}$$

   The RMS Johnson noise current in the detector (temperature $T_D$ = 4.2 K), on the other hand, is

$$\frac{\Delta I_{\rm rms}(\text{Johnson})}{\sqrt{\Delta f}} = \sqrt{\frac{kT_D}{R}} = 3.3\times10^{-15}\ \text{A Hz}^{-1/2} \quad . \tag{4}$$

   So, shot noise in the photocurrent dominates the Johnson noise in this case.

2. Noise in a random walk. *A friend of yours has just left your party, in a world-record state of drunkenness. She can't drive home, because you've wisely hidden her car keys, but her walking suffers from the following restriction: to hold herself up she leans her back against the wall of the building, so all she can do is stagger along in one dimension. Every lurching step she takes is one meter long. The directions of her steps are of course completely random; suppose the probability of any given step being in her right-hand direction is* q. *You wish to estimate where she's likely to be as a function of how many steps she's taken.*

   a. *Argue that the probability that she has traveled* N *steps to the right out of* n *total steps is governed by the binomial distribution,*

$$p_n(N) = \frac{n!}{N!(n-N)!}q^N(1-q)^{n-N} \quad .$$

If the probability for a step to the right is $q$, then the probability of $N$ steps to the right is $q^N$. Since she takes a step to the left whenever she doesn't take a step to the right, $N$ right steps corresponds to $n$ - $N$ left steps, and the probability of having taken this many left steps is $(1-q)^{n-N}$. The number of different ways one can get a *total* of $N$ right steps out of $n$ steps, if it doesn't matter which order the $N$ right steps come in, is $n!/(N!(n-N)!)$. The net probability, then, for getting $N$ right steps out of $n$ total steps is the product of these three terms, giving the binomial probability distribution.

b. *Derive an expression for her average distance from your door after n total steps. Hint: You will probably want to use the binomial theorem in this form:*

$$(x+y)^k = \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} x^j y^{k-j} \quad .$$

Let the +$x$-axis be to the right, and call the length of her steps $D$. For $N$ steps to the right, her position is $x = ND - (n-N)D = 2ND - nD$, so her *average* position is $\overline{x} = 2\overline{N}D - nD$, since everything on the right-hand side is a constant but $N$. In the present probability distribution, $\overline{N}$ is given by

$$\overline{N} = \sum_{N=0}^{n} N p_n(N) = \sum_{N=0}^{n} N \frac{n!}{N!(n-N)!} q^N r^{n-N} \quad , \tag{5}$$

where we have substituted $r = 1 - q$ to make the resemblance of this formula to the given form of the binomial theorem a little more obvious. Now we can use the derivative trick we've employed a couple of times in lecture. Noting that

$$Nq^N = q\frac{\partial}{\partial q} q^N \quad , \tag{6}$$

we can make this substitution in Equation (5) and reverse the order of differentiation and summation:

$$\begin{aligned} \overline{N} &= \sum_{N=0}^{n} \frac{n!}{N!(n-N)!} r^{n-N} q \frac{\partial}{\partial q} q^N \\ &= q\frac{\partial}{\partial q} \sum_{N=0}^{n} \frac{n!}{N!(n-N)!} q^N r^{n-N} \quad . \end{aligned} \tag{7}$$

This time the derivative stays partial; we do not want to operate on $r$'s implicit dependence upon $q$. Now invoke the binomial theorem, to get

$$\overline{N} = q\frac{\partial}{\partial q}\left((q+r)^n\right) = nq(q+r)^{n-1} = nq \tag{8}$$

(*now* remember $r$ = 1-$q$), so

$$\overline{x} = 2nqD - nD \quad . \tag{9}$$

  

c. *Derive an expression for the RMS deviation of her distance from your door, after* n *total steps.*

The variance in her position is

$$\begin{aligned}\overline{\Delta x^2} &= \overline{(x-\overline{x})^2} = \overline{(2ND-2nqD)^2} \\ &= 4D^2\left(\overline{N^2}+n^2q^2-2\overline{N}nq\right) \\ &= 4D^2\left(\overline{N^2}-n^2q^2\right) \quad .\end{aligned} \tag{10}$$

To evaluate $\overline{N^2}$, we use the derivative trick, this time in the form

$$N^2q^N = q\frac{\partial}{\partial q}\left(q\frac{\partial}{\partial q}q^N\right) \quad , \tag{11}$$

and the binomial theorem again:

$$\begin{aligned}\overline{N^2} &= \sum_{N=0}^{n} N^2\frac{n!}{N!(n-N)!}q^N r^{n-N} = \sum_{N=0}^{n}\frac{n!}{N!(n-N)!}r^{n-N}q\frac{\partial}{\partial q}q\frac{\partial}{\partial q}q^N \\ &= q\frac{\partial}{\partial q}q\frac{\partial}{\partial q}\sum_{N=0}^{n}\frac{n!}{N!(n-N)!}q^N r^{n-N} = q\frac{\partial}{\partial q}q\frac{\partial}{\partial q}(q+r)^n \\ &= q\frac{\partial}{\partial q}nq(q+r)^{n-1} = nq(q+r)^{n-1}+n(n-1)q^2(q+r)^{n-2} \\ &= nq+n(n-1)q^2 \quad .\end{aligned} \tag{12}$$

Thus

$$\overline{\Delta x^2} = 4D^2\left(nq-nq^2\right) \quad , \tag{13}$$

or

$$(\Delta x)_{\mathrm{rms}} = 2D\sqrt{nq(1-q)} \quad . \tag{14}$$

d. *Suppose she is equally likely to step to the left or the right. How far from your door is she likely to be after 120 steps? (Report this in the form* x±Δx.*) What is the probability that she'll be falling through your neighbor's open door, 40 meters to her right of yours, after 120 steps? Hint: remember Stirling's approximation for the factorial of a large number* n*:*

$$\ln(n!) \approx n\ln n - n + \frac{1}{2}\ln(2\pi n) \quad .$$

For this we have $q = 1/2$, $n = 120$ and $D = 1$ m, so plugging into Equations (9) and (14) gives $x \pm \Delta x = 0 \pm 11$ m. From this it appears that she's not very likely to be at $x = 40$ m; she would need to have taken $N = 80$ steps to the right (and 40 to the left), for which the probability is

  

$$p_{120}(80) = \frac{120!}{80!40!}\left(\frac{1}{2}\right)^{120} = 8.6\times10^{-5} \quad . \tag{15}$$

(8.6 fallings-in, in $10^5$ tries!)

Many calculators may have trouble with the arithmetic here, because of the large values of 120! and 80!. In such circumstances, Stirling's approximation for the logarithm of *n*! is useful. Since division is the same as subtraction of exponents,

$$\begin{aligned} \ln\left(\frac{120!}{80!40!}\right) &\approx 120\ln(120) - 120 + \frac{1}{2}\ln(240\pi) \\ &\quad -80\ln(80) + 80 - \frac{1}{2}\ln(160\pi) \\ &\quad -40\ln(40) + 40 - \frac{1}{2}\ln(80\pi) \\ &= 73.821 \quad , \end{aligned} \tag{16}$$

or

$$\frac{120!}{80!40!} \approx 1.14\times10^{32} \quad , \tag{17}$$

after which any calculator can do it.

e. *Suppose now that the weight of her backpack, slung over her right shoulder, leans her over enough that she is twice as likely to take a step to the right as to the left. Now how far from your door is she likely to be after 120 steps? (Report it in the form* x±Δx.*) What is the probability now that she'll fall through your neighbor's door on her* $120^{th}$ *step (if she hasn't already)?*

Now $q$ = 2/3, and plugging into Equations (9) and (14) again gives $x \pm \Delta x$ = 40±10 m. It looks like she'd be much more likely to have taken $N$ = 80 right steps and fall through the door at $x$ = 40 m this time, and indeed she is, but the probability is still a lot lower than many would expect:

$$p_{120}(80) = \frac{120!}{80!40!}\left(\frac{2}{3}\right)^{80}\left(\frac{1}{3}\right)^{40} = 0.077 \tag{18}$$

– that is, a 7.7% chance. Because of *noise* in her random walk, the probability of her being at $x$ = 40 m after 120 steps isn't all that close to 1 even though her average position in many hypothetical 120-step trials would be $x$ = 40 m.

  