Astronomy 203 Problem Set #8: Solutions

10 December 1999

1. A pressure-scanned Fabry-Perot interferometer. *At normal atmospheric pressure and room temperature, the index of refraction of* CO_2 *is 1.0045. At the same temperature and a pressure of 4 atmospheres, the index is 1.0180. For constant temperature, the index varies linearly with pressure between these extremes. Using a few complete sentences, suggest a way of using this effect to tune a Fabry-Perot interferometer. For a pressure varying between 1-4 atmospheres, and an interferometer with spacing 0.3 cm and finesse* Q = 30, *operating at a wavelength 0.55* μ*m, what range of wavelengths is covered by the scan? Which Fabry-Perot order is used? How many FWHM resolution elements are contained in the scan? (Many high-resolution, visible-wavelength Fabry-Perot spectrometers employ this principle.)*

   Because constructive interference in a Fabry-Perot is obtained for $2nd\cos\theta = m\lambda$, increasing the index of refraction n does the same thing as increasing the mirror separation d – that is, it moves the orders to longer wavelengths. Thus one can scan the interferometer by varying the pressure instead of the distance. For normal incidence ($\theta = 0$), the change in order wavelength $\delta\lambda$ produced by a change in index δn is given by

   $$\delta\lambda = \frac{2d\delta n}{m} = \lambda\delta n = 0.0074\ \mu\mathrm{m} \tag{1}$$

   for the wavelength and index change given above. This effect is very useful for high-resolution spectroscopy even though 0.0074 µm doesn't seem like the spectrometer covers much of the spectrum. At the given wavelength and mirror separation and at atmospheric pressure, the interferometer is in $m = 2d\delta n/\delta\lambda = 10{,}900^{\mathrm{th}}$ order, and the FWHM resolution is $\Delta\lambda = \lambda/mQ = 0.0000017$ µm, so the pressure scan up to 4 atm moves the order in use by about 4400 spectral resolution elements.

2. Beam size and spectral resolution of a Fabry-Perot. *An incoherent detector looks through a Fabry-Perot interferometer at normal incidence, with a beam of small angular radius* θ.

   a. *Show that the detector is therefore sensitive to a range of wavelengths, varying from* $\lambda = 2nd/m$ *to* $\lambda' = 2nd(1 - \theta^2/2)/m$*, and that a wavelength resolution element can therefore be no smaller than*

   $$\Delta\lambda = \lambda\frac{\theta^2}{2} \quad .$$

   We will refer to this result as the beam-divergence limit to the spectral resolution of a Fabry-Perot interferometer.

   Constructive interference takes place when $4\pi nd\cos\theta'/\lambda = 2\pi m$, where θ' is the incidence angle for light between the reflectors, and where the refractive index is n. If a range of incidence angles is used to illuminate the Fabry-Perot, therefore, constructive interference will be obtained for a range of wavelengths for a given order m and a fixed reflector separation d. If the incidence angle is small ($\cos\theta' \approx 1 - \theta'^2/2$), this range of wavelengths $\Delta\lambda$ transmitted by the Fabry-Perot is given by

   $$\Delta\lambda = \frac{2nd}{m} - \frac{2nd\left(1-\frac{\theta'^2}{2}\right)}{m} = \frac{2nd}{m}\frac{\theta'^2}{2} = \lambda\frac{\theta'^2}{2} \quad . \tag{2}$$

Assuming the index of refraction to be the same inside and outside the Fabry-Perot, $\theta' = \theta$. Note that if the index between the reflectors is larger than it is outside, $\theta' < \theta$, and $\Delta\lambda$ is smaller for the same beam angular radius.

b. *Suppose you wanted to have the beam be 0.1 radian (5.7°) in radius. For a Fabry-Perot with a finesse of 20, what is the highest order number you can use before the beam-divergence limitation on the spectral resolution is equal to the reflectance-limited resolution?*

In other words: in which order is the transmitted range of wavelengths equal to the FWHM reflectance-limited width? The former is given by $\Delta\lambda = \lambda\theta^2/2$, the latter by $\Delta\lambda = \lambda/mQ$, so they're equal for $m = 2/Q\theta^2$. With the parameters we're given here, $m = 10$.

c. *Suppose further that you really need better spectral resolution than that. Suggest an optical configuration for the Fabry-Perot that will overcome the beam-divergence limitation.*

If we need higher resolution, we can get it by using a higher order of the Fabry-Perot – but not in the present optical configuration, since for any order $m > 10$ the transmitted range of wavelengths is dominated by the beam-divergence limit. To reduce this effect, one needs to use lenses to re-focus the beam through the Fabry-Perot so as to give a smaller angular radius (see Figure 1). The best job is done by collimating the beam and placing the Fabry-Perot in the collimated part of the optics. It doesn't need absolutely to be collimated (angular spread = 0), though; we just need for $\Delta\lambda$ from the beam divergence to be much smaller than $\Delta\lambda$ from the reflectance, or $\theta^2 << 2/mQ$.

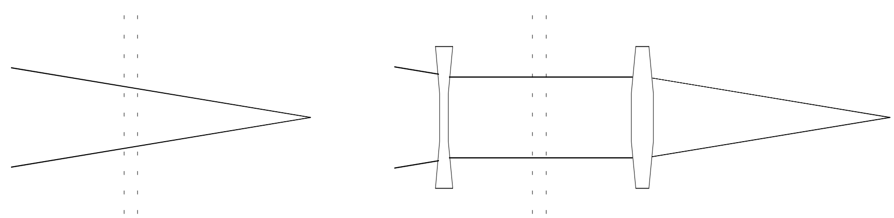

Figure 1: the Fabry-Perot on the right can be used to obtain higher spectral resolution than the one on the left, since the beam it transmits has a smaller angular radius.

3. Diffraction grating measurements of the sodium D-lines (λ = 0.58959, 0.58900 μm).

a. *A diffraction grating has 10^4 rulings uniformly spaced over 2.5 cm. It is illuminated by yellow light from a low-pressure sodium-vapor lamp, at normal incidence. At what angles will the first-order maxima occur for these lines?*

For an incidence angle of zero, $a\sin\theta_m = m\lambda$. Here $a = 2.5$ μm , and the first-order angles work out to $\theta_m = 13.641°$ for the 0.58959 μm line and $\theta_m = 13.627°$ for the 0.58900 μm line.

b. *How many rulings must a diffraction grating have in order barely to resolve them in third order?*

The resolution of a grating in m^{th} order is $\Delta\lambda/\lambda = 1/mN$, and if the lines are barely resolved, then $\Delta\lambda$ is equal to the difference between the wavelengths of the two lines, or 0.00059 μm. Thus $N = \lambda/m\Delta\lambda = 333$ rulings.

c. *In a particular grating the D-lines are viewed in third order at 80° to the normal and are barely resolved. How far apart are the grating rulings?*

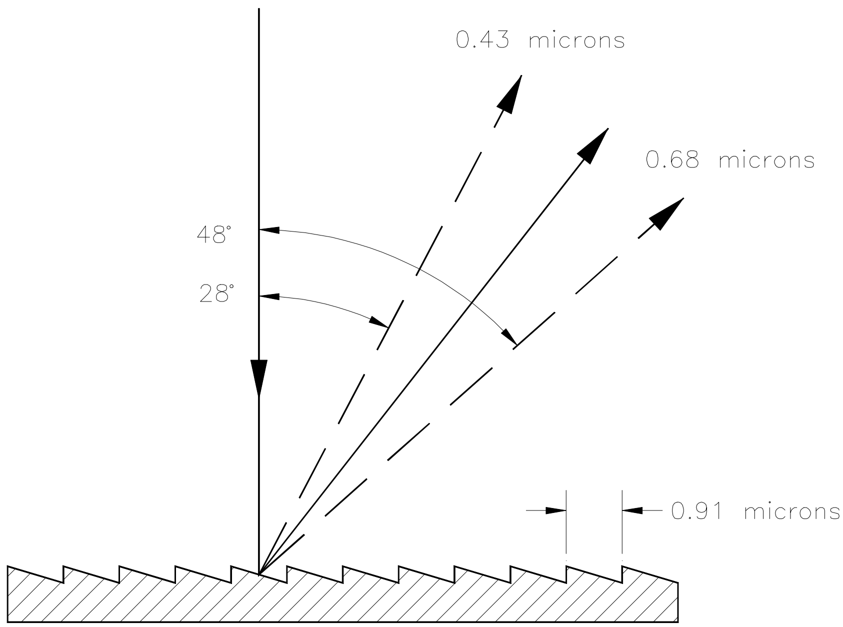

Figure 2: a grating configuration that spreads the first order visible spectrum through 20°.

For normal incidence, $a = m\lambda/\sin\theta_m = 1.8\ \mu\text{m}$.

4. Grating spectrometer design. *Taking the limits of the visible spectrum to be λ = 0.43 - 0.68 μm, design a grating that will spread the first order spectrum through an angular range of 20°. Use any incidence angle you like. Report the incidence angle, the range of diffracted angles, and the proper blaze angle.*

There are as many correct ways to do this as there are angles of incidence. Let's do normal incidence first. What we want is for the shortest wavelength of the range, say $\lambda_S = a\sin\theta_m$, to wind up $\Delta\theta = 20°$ away from the longest wavelength, $\lambda_L = a\sin(\theta_m+\Delta\theta)$. We just need to solve these two equations for a and θ_m. The latter can be obtained from the ratio of the equations:

$$\frac{\lambda_L}{\lambda_S} = \frac{\sin(\theta_m + \Delta\theta)}{\sin\theta_m} = \frac{\sin\theta_m \cos\Delta\theta + \cos\theta_m \sin\Delta\theta}{\sin\theta_m} = \cos\Delta\theta + \frac{\sin\Delta\theta}{\tan\theta_m} \quad , \tag{3}$$

where we have used a trig identity, $\sin(A+B) = \sin A\cos B+\cos A\sin B$. Solving this for θ_m, we get

$$\theta_m = \arctan\left(\frac{\sin\Delta\theta}{\frac{\lambda_L}{\lambda_S} - \cos\Delta\theta}\right) = 28° \quad . \tag{4}$$

So λ_S=0.43 µm comes out at θ_m=28°, and λ_L=0.68 µm comes out at θ_m=48°. Either of these pairs determines $a = \lambda/\sin\theta_m$ = 0.91 µm. The blaze angle should be such that the ruling normals bisect the incident and outgoing angles; one would probably therefore blaze the grating for the central angle of θ_m=38°, which gives a blaze angle $\gamma = (\theta_m-\theta_i)/2$ = 19°. A diagram of the finished grating configuration is shown in Figure 2.

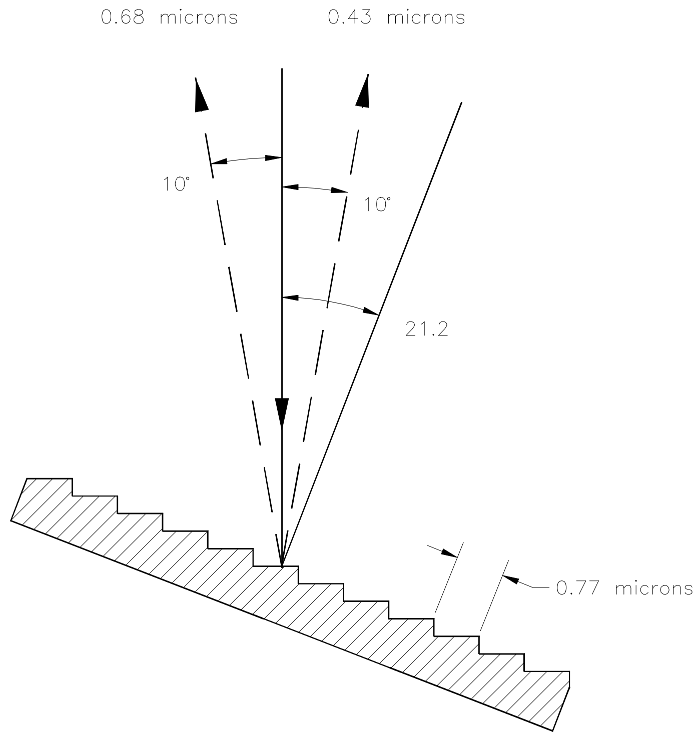

Figure 3: another grating configuration (Littrow) that spreads the first order visible spectrum through 20°.

We might have tried to use a Littrow configuration ($\theta_i = -\theta_m$) instead. If we did, and tried to get the extremes of the wavelength range to come out ±10° from the optical axis, we would have wound up with an incident angle of 21.2°, a blaze angle also of 21.2°, and a ruling spacing of 0.77 µm. See Figure 3 for a diagram of this alternative.

5. The entrance slit of a grating spectrometer. *Consider a telescope with a grating spectrometer, as shown schematically in Figure 4. The telescope has diameter* D *and focal length* F, *the grating-spectrometer collimating mirror has diameter* d *and focal length* f, *and the light can enter the spectrometer through a slit of width* x. *The spectrometer is in Littrow mode* $(\theta_m = -\theta_i)$.

   a. *If x is not zero, the "collimated" light hitting the grating has some angular spread. Explain why, and show that the angular spread is given by* $\Delta\theta = x / f$ *, and thus that the resolution of the spectrometer is*

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{\lambda}\frac{x}{f}\frac{a\cos\theta_m}{m}$$

   – *that is, you can make the resolution better by making x smaller.*

   A wavelength interval in diffracted light, $\Delta\lambda$, is related to its range of diffraction angles $\Delta\theta_m$ by the dispersion:

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{\lambda}\frac{\Delta\theta_m}{\left(\dfrac{d\theta_m}{d\lambda}\right)} \quad . \tag{5}$$

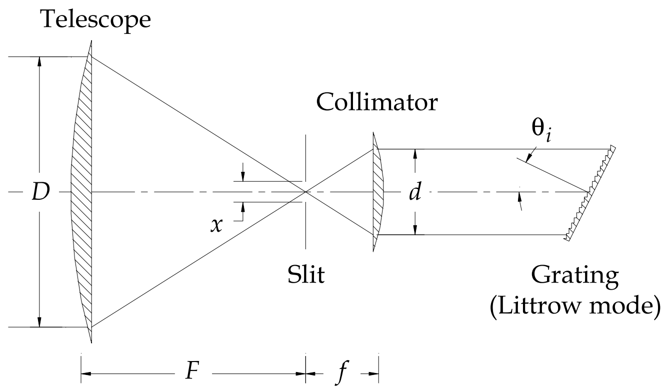

Figure 4: optical setup for Problem 5.

The collimator/camera has a plate scale given by $1/f$, so the range of angles $\Delta\theta_i$ corresponding to a range of distances x in the focal plane is simply $\Delta\theta_i = x/f$. For a given wavelength λ the range in diffracted angles $\Delta\theta_m$ produced by the range in incidence angles is obtained as follows:

$$m\lambda = a\left(\sin\theta_m - \sin\theta_i\right) \qquad \Rightarrow \qquad \theta_m = \arcsin\left(\sin\theta_i + \frac{m\lambda}{a}\right) \quad ,$$

$$\frac{d\theta_m}{d\theta_i} = \frac{\cos\theta_i}{\sqrt{1-\left(\sin\theta_i + \dfrac{m\lambda}{a}\right)^2}} \quad . \tag{6}$$

In particular, at the wavelength λ_m for which $\theta_i = -\theta_m$, $m\lambda_m/a = \sin\theta_m - \sin\theta_i = -2\sin\theta_i$, so

$$\frac{d\theta_m}{d\theta_i} = \frac{\cos\theta_i}{\sqrt{1-\left(-\sin\theta_i\right)^2}} = 1 \qquad \Rightarrow \qquad \Delta\theta_m = \frac{d\theta_m}{d\theta_i}\Delta\theta_i = \Delta\theta_i = \frac{x}{f} \quad . \tag{7}$$

But θ_i doesn't depend upon λ, so the dispersion $d\theta_m/d\lambda$ is still given by Equation 27.8:

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{\lambda}\frac{\Delta\theta_m}{\left(\dfrac{d\theta_m}{d\lambda}\right)} = \frac{1}{\lambda}\frac{x}{f}\frac{a\cos\theta_m}{m} \quad . \tag{8}$$

This applies to wavelengths near λ_m, as assumed in going from Equation (6) to Equation (7).

b. *Of course, diffraction prevents you from making x arbitrarily small. Show that the smallest x is allowed to be is* $1.2f\lambda/d$, *and therefore that the smallest* $\Delta\lambda/\lambda$ *can be is*

$$\frac{\Delta\lambda}{\lambda} = \frac{1.2}{mN} \quad .$$

Since the beam has a finite diameter d, and is *circular*, the FWHM angular spread from diffraction is $\Delta\theta_m \cong 1.2\lambda/d$, no matter what goes on in the focal plane. For a plate scale of $1/f$ in the focal plane, the diffraction spot has FWHM diameter $x \cong 1.2\lambda f/d$. Transmission of light to the grating, and thence to the focal plane, from a slit smaller than this does not reduce the angular range in the way suggested by the result from part a — that just lets less light through. Thus the smallest $\Delta\lambda$ can be is that given by a diffraction-limited slit width, $x \cong 1.2\lambda f / d$:

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{\lambda}\frac{1.2\lambda f}{fd}\frac{a\cos\theta_m}{m} = \frac{1.2}{mN} \quad ,$$

since the length of the grating is $Na = d / \cos\theta_i = d / \cos\theta_m$.

c. *Suppose you want the slit to match the image size from the telescope's beam, an angle ϕ in size. Find an expression for the resolution, in terms of ϕ, D, and d. (This, and the previous result, show how one "matches" a grating spectrometer to a telescope.)*

The telescope has plate scale $1/F$, so an angular range ϕ corresponds to a width $F\phi$ in the focal plane. By the same token, the focal plane width $F\phi$ corresponds to an angular spread $\Delta\theta_i = F\phi / f$ in the collimated beam. Thus, from part a,

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{\lambda}\frac{F\phi}{f}\frac{a\cos\theta_m}{m} = \frac{1}{\lambda}\frac{D\phi}{d}\frac{a\cos\theta_m}{m} \quad ,$$

where we have used the relation $D/d = F/f$, obvious from the geometry in Figure 4.

