3. Lecture, 9 September 1999

3.1 Conic sections

Most of the mirrors used in telescopes have geometrical figures that one can generate by rotation of two-dimensional conic sections about their axes of symmetry. Here we summarize the properties of these curves; you have presumably seen the details behind the summary in an analytical geometry class.

1. The *circle*, seen in Figure 3.1 (although this seems trivial), rotates to a sphere. It is described by

$$x^2 + y^2 = a^2 \quad . \tag{3.1}$$

The radius of curvature throughout the curve is $r = a$, of course. One can use this obvious fact to verify the general expression for the *curvature* (reciprocal of the radius of curvature) of a two-dimensional curve:

$$\kappa = \frac{1}{r} = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}} = \frac{\left|\frac{d^2x}{dy^2}\right|}{\left(1+\left(\frac{dx}{dy}\right)^2\right)^{3/2}} \quad . \tag{3.2}$$

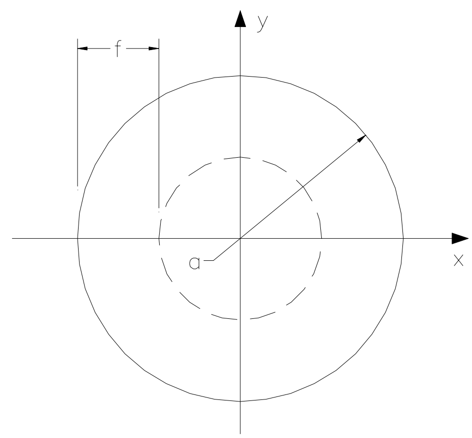

Figure 3.1

The focal length is $f = a/2$, as we have discussed. Since the figure is symmetrical, the sphere has a spherical paraxial focal surface, as indicated in Figure 3.1, instead of a discrete set of foci like the other conic sections.

The circle is a special case of the

2. *ellipse*, which rotates to an ellipsoid (Figure 3.2). It is described analytically by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad . \tag{3.3}$$

For $a > b$, there are two foci on the x axis, a distance $c = \sqrt{a^2 - b^2}$ either side of the origin. The long axis (length $2a$), which intersects the curve at its two apices and passes through the foci, is called the *major axis*; this is the symmetry axis used to generate ellipsoids by rotation. At each apex, the radius of curvature is

$$r = \frac{b^2}{a} \quad . \tag{3.4}$$

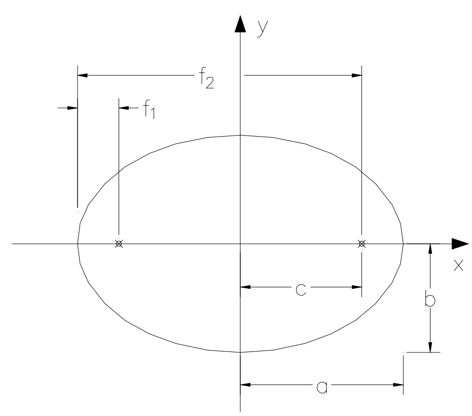

Figure 3.2.

a and b are respectively called the *semimajor* and *semiminor* axes. It is convenient to define the *eccentricity* ε, which is the ratio of the focal distance c to the semimajor axis length a:

$$\varepsilon = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} \qquad (0 \le \varepsilon < 1) \quad ; \tag{3.5}$$

a circle is an ellipse with zero eccentricity. Ellipsoidal mirrors are often (but not always) used *on axis*, meaning that incident light is nearly parallel to the major axis and only a apex region is present; in this situation the ellipsoid has two focal lengths (the distances from the apex to each focus), given by

$$f = a \pm c \quad . \tag{3.6}$$

When they are used off axis – that is, when a section away from the apex is used – the relevant focal lengths are the distances from the center of the section to each of the foci.

3. The *parabola*, seen in Figure 3.3, rotates to a paraboloid. It is described by

$$y = ax^2 \quad . \tag{3.7}$$

At its apex the radius of curvature is

$$r = \frac{1}{2a} \quad . \tag{3.8}$$

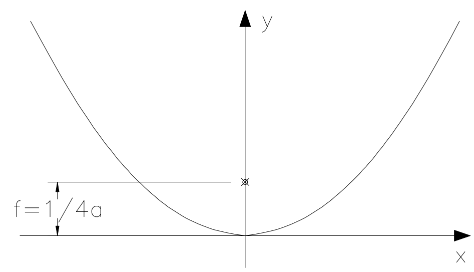

Figure 3.3.

It has only one focus, a distance

$$f = \frac{1}{4a} \tag{3.9}$$

from the apex. This is the focal length when the apex region of the mirror is illuminated on axis, but off-axis sections of parabolas are frequently used as well. Parabolas have $\varepsilon = 1$.

4. Finally, the *hyperbola*, graphed in Figure 3.4, is described by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad . \tag{3.10}$$

When rotated about the major axis containing the apices of the two branches, a hyperboloid is generated. Like the ellipse, the hyperbola has two foci; these lie along the major axis a distance $c = \sqrt{a^2 + b^2}$ either side of the center. The radius of curvature at either apex is

$$r = \frac{b^2}{a} \quad , \tag{3.11}$$

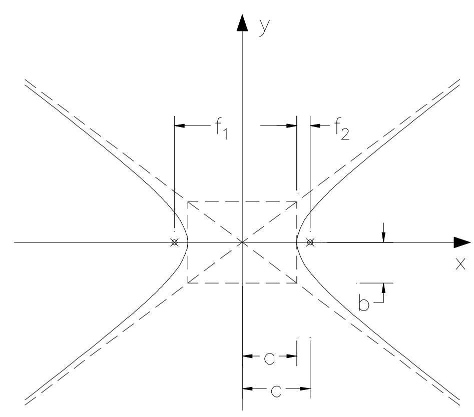

Figure 3.4.

the eccentricity is

$$\varepsilon = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} > 1 \quad , \tag{3.12}$$

and when used on axis the two focal lengths are

$$f = c \pm a \quad . \tag{3.13}$$

The apex regions of conic sections are the most common mirrors in telescopes and astronomical instruments. In terms of the apex curvature $\kappa = 1/r$ (see figure 3.5), given for the individual conics by equations 3.4, 3.8 and 3.11, and of the eccentricity ε, the apex regions of all four conic sections can be described by a single formula:

$$y^2 - \frac{2}{\kappa}x + \left(1-\varepsilon^2\right)x^2 = 0 \quad , \tag{3.13}$$

or, solved for y,

$$y = \frac{\kappa x^2}{1+\left[1-\kappa^2 x^2\left(1-\varepsilon^2\right)\right]^{1/2}} \quad , \tag{3.14}$$

The quantity $1-\varepsilon^2$ appears frequently in the equations of optics and classical mechanics; it is called the *conic constant*.

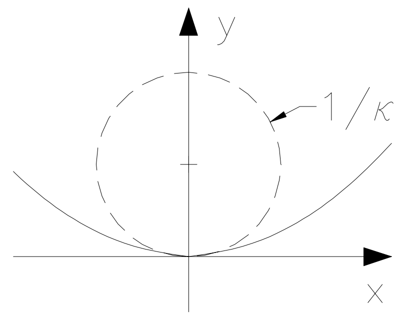

Figure 3.5: The apex region of a conic section.

3.2 Conic section mirrors

Why are conic section mirrors used in astronomical instruments? Because

1. Paraboloidal mirrors reflect all on-axis, parallel light to their focus. (That's why that point is *called* the focus.)

1. Ellipsoidal or hyperboloidal mirrors reflect light that would converge on one focus, and make it instead converge on the other focus.

Because we will use the first of these assertions to test our upcoming analytical treatment of ray tracing, we will prove it in the following. A proof of the second statement can be obtained similarly.

Consider the parabola $y = x^2/4f$; its focus lies at $y = f$. Consider, furthermore, a ray incident from point *O* along the line $x = x_0$; construct a line from the focus *F* to the intersection *A* of the line and the parabola, as shown in Figure 3.6. Construct also the perpendicular to the parabola at point *A*; this line intersects the y axis at point *C*. If the "incidence angle" *OAC* and the "reflection angle" *CAF* are congruent, and if this is true independent of the choice of x_0, then all rays incident parallel to the y axis will reflect through the focus.

Claim: Angle *OAC* and angle *CAF* (Figure 3.6) are congruent, independent of x_0.

Proof: This statement amounts to the claim that the triangle *AFC* is isosceles. This is easy to prove because we can determine the lengths of *AF* and *CA*. The former is simply

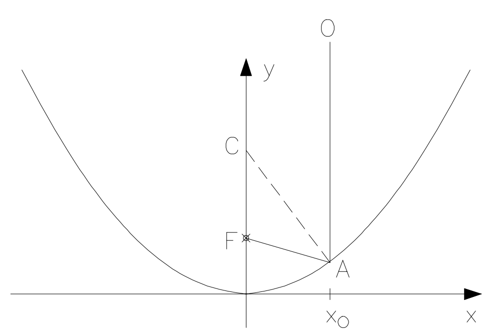

Figure 3.6: proof that paraboloids reflect on-axis rays through their foci.

$$AF = \sqrt{x_0^2 + \left(f - \frac{x_0^2}{4f}\right)^2} = \sqrt{x_0^2 + f^2 + \left(\frac{x_0^2}{4f}\right)^2 - \frac{x_0^2}{2}}$$
$$= \sqrt{\left(f + \frac{x_0^2}{4f}\right)^2} = f + \frac{x_0^2}{4f} \quad . \tag{3.15}$$

For the latter, we need the equation of the line through *AC*. The slope of the parabola at *A* is

$$\frac{dy}{dx}(x_0) = \frac{x_0}{2f} \quad , \tag{3.16}$$

so *AC* itself has slope $-2f / x_0$, and the line is described by

$$\left(y - \frac{x_0^2}{4f}\right) = -\frac{2f}{x_0}(x - x_0) \quad . \tag{3.17}$$

The y-intercept of this line – the position of point *C* – is at

$$y_0 = \frac{x_0^2}{4f} + 2f \quad , \tag{3.18}$$

and thus

$$CF = y_0 - f = f + \frac{x_0^2}{4f} = AF \quad , \tag{3.19}$$

independent of the choice of x_0 and *f*. Therefore *OA* and *AF* are inclined by the same angle with respect to the perpendicular to the parabola at *A;* all incident rays parallel to the symmetry axis of a paraboloidal mirror are reflected through the focus. (Q.E.D.)

3.3 Conic-section mirrors and the thin-lens equation

Within the confines of paraxial rays, one can use the thin lens equation with conics, just as with thin spherical mirrors, for a first-order determination of the locations of images. Paraboloids have only one focal point, and can be used straight away with the thin lens equation, noting only that the focal length of interest is the distance from the focal point to the center of the section of the paraboloid in use; in general this differs from the on-axis focal length, as shown in Figure 3.7.

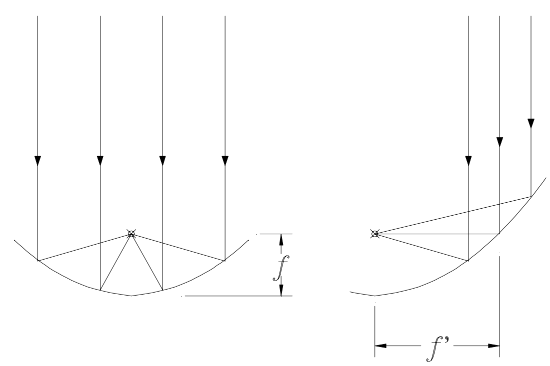

Figure 3.7: focal lengths of on-axis (left) and off-axis (right) segments of the same paraboloidal mirror figure.

The situation is slightly different for ellipsoids and hyperboloids, which have two foci, like lenses, but with different focal lengths, unlike lenses. They can, however, be represented as two thin lenses in contact, with focal lengths equal to their two focal lengths, $f = a \pm c$ (ellipsoids) or $f = c \pm a$ (hyperboloids), for an equivalent focal length (see Equation 3.15) given by

$$f = \frac{f_1 f_2}{f_1 + f_2} \quad . \tag{3.20}$$

There is, however, one confusing difference in the case of hyperboloids: they are exempt from the normal sign convention. For these mirrors, real and virtual images occur on opposite sides of the mirror for a given object side (like lenses), rather than on the same side (like the other conic-section mirrors, including spheres). This is probably best illustrated by an example:

Example 3.1

Consider a Cassegrain telescope, consisting of a concave paraboloid mirror and a convex hyperboloid. The near focus of the hyperboloid is coincident with the focal point of the paraboloid. Where is the final focal point of the system? To be concrete, take the focal length of the paraboloid to be 20 cm, and those of the hyperboloid to be 4 and 16 cm.

The situation is sketched in Figure 3.8. The first image, formed by the paraboloid, for a very distant object, is given by

$$\frac{1}{o_1} + \frac{1}{i_1} = \frac{1}{f}$$
$$\frac{1}{i_1} = \qquad \Rightarrow \quad i_1 = f = 20 \text{ cm} \;;$$

The hyperboloid's apex is placed 4 cm in front of this point, for which the usual sign convention would have $o_2 = -4$ cm; however, because the hyperboloid's foci are separated by the mirror we must take

$$o_2 = i_1 - 16 \text{ cm} = +4 \text{ cm} \, ,$$

so that

$$\frac{1}{o_2} + \frac{1}{i_2} = \frac{1}{f_{hyperb}} = \frac{f_1 + f_2}{f_1 f_2} \quad .$$

Now, o_2 happens to be equal to f_1, so

$$\frac{1}{f_1} + \frac{1}{i_2} = \frac{f_1 + f_2}{f_1 f_2} = \frac{1}{f_1} + \frac{1}{f_2} \quad \Rightarrow \quad i_2 = f_2 = +16 \text{ cm}$$

on the opposite side of the hyperboloid from the first image, which places it on the apex of the paraboloid.

Note that this works the same, with no sign-convention exceptions, if the hyperboloid is replaced by an ellipsoid with the same focal lengths. If you find hyperboloids confusing, it's OK to replace them conceptually by the equivalent ellipsoid. Bear in mind, though, that although they are the same in paraxial theory (except for the sign-convention exception), they're not the same in detail, as we'll see when we consider geometrical aberrations.

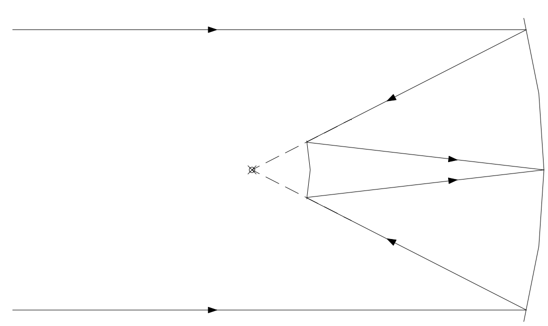

Figure 3.8: the Cassegrain telescope in Example 3.1.

