13. Lecture, 14 October 1999

13.1 Example: diffraction from a circular aperture

Most telescopes have circular apertures, so the application of the Kirchhoff integral to diffraction from such apertures is of particular interest. Let us start with a plane wave incident on a circular hole with radius a in an otherwise opaque screen, and calculate the distribution of the intensity of light on a screen a distance $R >> a$ away, as in Figure 13.1. The amplitude of the near field we consider to be constant, so that $E_N(x',y',t) = E_{N0}e^{-i\omega t}$. With the definitions

$$\begin{aligned} x' &= r'\cos\phi' & X &= q\cos\Phi & \kappa_x &\cong \kappa\frac{X}{r} = \frac{\kappa q\cos\Phi}{r} \\ y' &= r'\sin\phi' & Y &= q\sin\Phi & \kappa_y &\cong \frac{\kappa q\sin\Phi}{r} \quad , \\ da' &= r'dr'd\phi' & & & & \end{aligned} \tag{13.1}$$

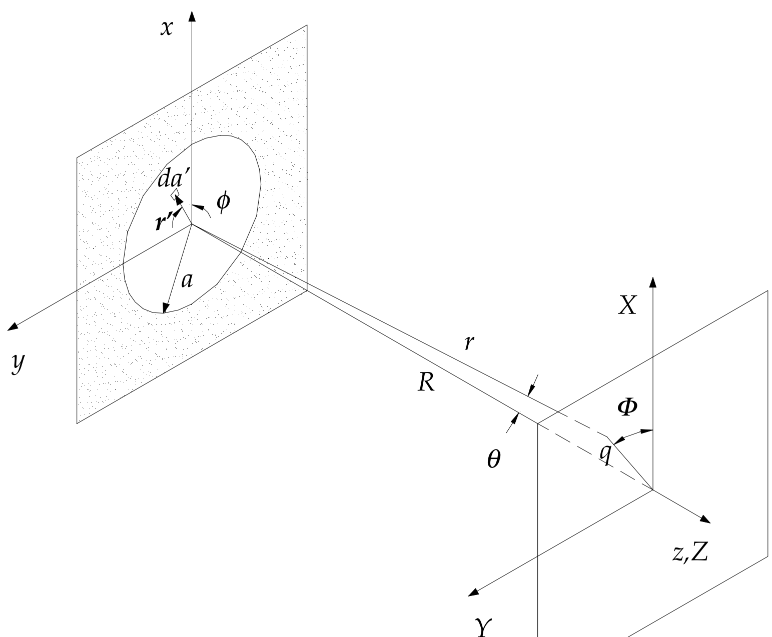

Figure 13.1: geometry of circular aperture and distant screen for calculation of diffraction of plane wave incident normally on the aperture. The "screen" axes X and Y are parallel to the "aperture" axes x and y.

the far field, at point q,Φ on the screen, is

$$\begin{aligned} E_F\left(\kappa_x,\kappa_y,t\right) &= \frac{e^{i\kappa r}}{\lambda r}\int_0^a dr'r'\int_0^{2\pi} d\phi' E_{N0}e^{-i\omega t}\exp\left(-\frac{i\kappa r'q}{r}\left(\cos\phi'\cos\Phi + \sin\phi'\sin\Phi\right)\right) \\ &= \frac{E_{N0}e^{i(\kappa r-\omega t)}}{\lambda r}\int_0^a dr'r'\int_0^{2\pi} d\phi'\exp\left(-\frac{i\kappa r'q}{r}\cos\left(\phi'-\Phi\right)\right) \quad , \end{aligned} \tag{13.2}$$

where the trig identity $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ has been used in the last step. The aperture is symmetrical about the *z* axis, so we expect that the answer will be independent of the "screen" azimuthal coordinate Φ; without loss of generality, then, we can take $\Phi = 0$. This leaves the following integral over ϕ':

$$I=\int_0^{2\pi} d\phi' \exp\left(-\frac{i\kappa r' q}{r}\cos\phi'\right) \quad . \tag{13.3}$$

Integrals of this form crop up frequently in physics, whenever there is axial symmetry. They cannot, however, be expressed in terms of the usual elementary functions such as the trig functions or exponentials, but instead are elementary functions themselves; this is a *Bessel function*.

The Bessel function of the first kind, of order *m*, has this integral representation:

$$J_m(u)=\frac{i^{-m}}{2\pi}\int_0^{2\pi} e^{i(mv+u\cos v)}dv \quad . \tag{13.4}$$

The first few of these are plotted in Figure 13.2. Note that the even-order Bessel functions are even functions and the odd-order ones are odd functions; our integral, Equation 13.3, therefore resembles a first-kind Bessel function of order zero:

$$\begin{aligned} &J_0(-u)=J_0(u)=\frac{1}{2\pi}\int_0^{2\pi} e^{iu\cos v}dv \quad ; \\ &I=2\pi J_0\left(\frac{\kappa r' q}{r}\right) \quad . \end{aligned} \tag{13.5}$$

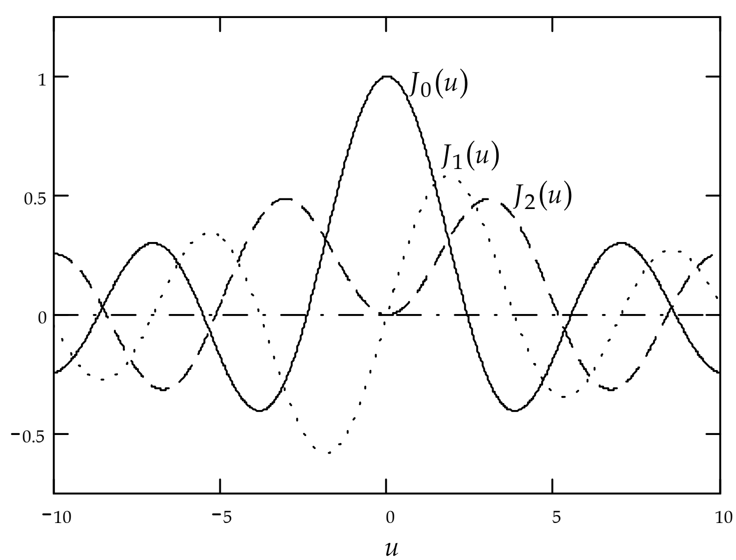

Figure 13.2: The first three of the first-kind Bessel functions.

Bessel functions are "oscillating" functions, but their zeros are not evenly spaced along the horizontal axis; this keeps them from being expressible as linear combinations of just a few sines and cosines. Some of the zeroes are listed in Table 13.1. One of these is listed in boldface – it will turn out to be important later.

J_0	J_1	J_2
2.405	0	0
5.520	**3.832**	5.136
8.654	7.016	8.417
11.792	10.174	11.620
14.931	13.324	14.796

Table 13.1: the first five zeroes of the Bessel functions plotted in Figure 13.2.

We thus use Equation 13.5 in Equation 13.3, and in turn in Equation 13.2, to produce

$$E_F(q,t) = \frac{2\pi E_{N0} e^{i(\kappa r - \omega t)}}{\lambda r} \int_0^a dr' r' J_0\left(\frac{\kappa q r'}{r}\right) = \frac{2\pi E_{N0} e^{i(\kappa r - \omega t)}}{\lambda r} \left(\frac{r}{\kappa q}\right)^2 \int_0^{\kappa a q / r} v J_0(v) dv \quad . \tag{13.6}$$

Integration of J_0 is facilitated by a recurrence relation between Bessel functions of different order:

$$\begin{aligned} \frac{d}{du}\left[u^m J_m(u)\right] &= u^m J_{m-1}(u) \quad , \\ \text{or} \qquad u^m J_m(u) &= \int_0^u v^m J_{m-1}(v) dv \quad . \end{aligned} \tag{13.7}$$

We can use the latter formula for $m = 1$ in Equation 13.6 to obtain

$$\begin{aligned} E_F(q,t) &= \frac{2\pi E_{N0} e^{i(\kappa r - \omega t)}}{\lambda r} \left(\frac{r}{\kappa q}\right)^2 \left(\frac{\kappa a q}{r}\right) J_1\left(\frac{\kappa a q}{r}\right) \\ &= \frac{\pi a^2 E_{N0} e^{i(\kappa r - \omega t)}}{\lambda r} \frac{r}{\kappa a q} 2 J_1\left(\frac{\kappa a q}{r}\right) \quad , \\ \text{or} \qquad E_F(\theta,t) &= \frac{E_{N0} A e^{i(\kappa r - \omega t)}}{\lambda r} \frac{2 J_1(\kappa a \theta)}{\kappa a \theta} \quad , \end{aligned} \tag{13.8}$$

where we have substituted $A = \pi a^2$ and $\theta = q / r$. The intensity on the screen becomes

$$I_F(\kappa a \theta) = \frac{c}{8\pi} E_F^*(\theta,t) E_F(\theta,t) = \frac{c E_{N0}^2 A^2}{8\pi \lambda^2 r^2} \left[\frac{2 J_1(\kappa a \theta)}{\kappa a \theta}\right]^2 \quad . \tag{13.9}$$

in cgs units. Let us rewrite this expression in terms of the intensity at $\kappa a \theta = 0$, which as you may suspect will turn out to be the *peak* intensity. Immediately we see a problem with evaluation of Equation 13.9 here;

the last factor is indeterminate at $\kappa a\theta = 0$. To proceed, note first that $J_0(0) = 1$ and that $J_1(0) = 0$ (see Figure 13.2 and Table 13.1), and that the recurrence relation, Equation 13.7, can be written for the first two first-kind Bessel functions as

$$J_0(u) = \frac{d}{du}J_1(u) + \frac{J_1(u)}{u} \quad . \tag{13.10}$$

Evaluation of this expression at $u = 0$ is complicated by the indeterminacy of the last term; we must write it as

$$\begin{aligned} 1 &= \frac{dJ_1}{du}(0) + \lim_{u\to 0}\frac{J_1(u)}{u} \\ &= \frac{dJ_1}{du}(0) + \lim_{u\to 0}\frac{\frac{dJ_1}{du}(u)}{1} \quad \text{(l'Hôpital's rule)} \\ &= 2\frac{dJ_1}{du}(0) \quad , \end{aligned} \tag{13.11}$$

or

$$\frac{dJ_1}{du}(0) = \frac{1}{2} \quad , \tag{13.12}$$

which, replaced in the first of Equations 13.11, gives

$$\lim_{u\to 0}\frac{J_1(u)}{u} = \frac{1}{2} \quad . \tag{13.13}$$

We can use this in Equation 13.9 to get

$$I_F(0) = \frac{cE_{N0}^2A^2}{8\pi\lambda^2r^2}\left[2\frac{1}{2}\right]^2 = \frac{cE_{N0}^2A^2}{8\pi\lambda^2r^2} \quad , \tag{13.14}$$

and

$$I_F(\kappa a\theta) = I_F(0)\left[\frac{2J_1(\kappa a\theta)}{\kappa a\theta}\right]^2 \quad . \tag{13.15}$$

Because J_1 also has zeroes at finite values of $\kappa a\theta$, $I_F(\kappa a\theta)$ has a set of concentric rings for which the intensity is zero (*dark* rings) The first of these lies at $\kappa a\theta_1 = 3.832$ (see Table 13.1), or

$$\theta_1 = \frac{3.832}{\kappa a} = \frac{3.832}{\pi}\frac{\lambda}{2a} = 1.22\frac{\lambda}{D} \quad , \tag{13.16}$$

where D is the diameter of the aperture – a familiar result. Equation 13.15 is plotted in various ways in Figure 13.3 and Figure 13.4.

George Airy, Astronomer Royal of England during the 1840s, first did the calculation we just completed; the intensity distribution of Equation 13.15 is usually called the *Airy pattern*, and the central maximum within the first dark ring is called the *Airy disc*.

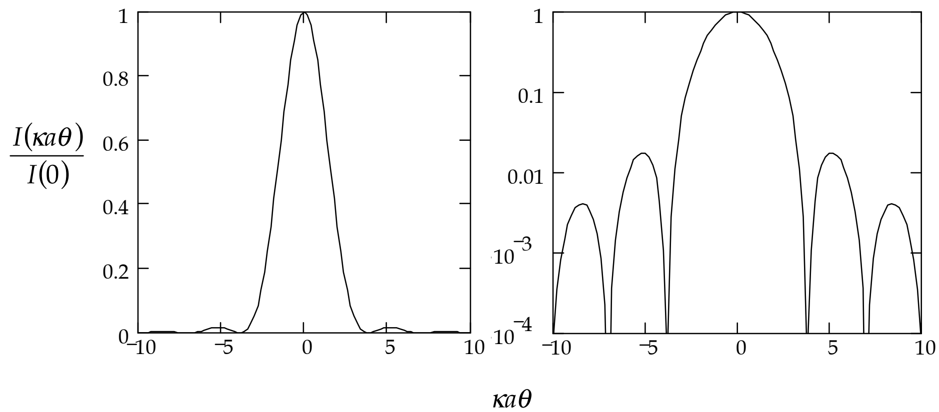

Figure 13.3: plots of the intensity given by Equation 13.15, on linear (left) and logarithmic (right) scales.

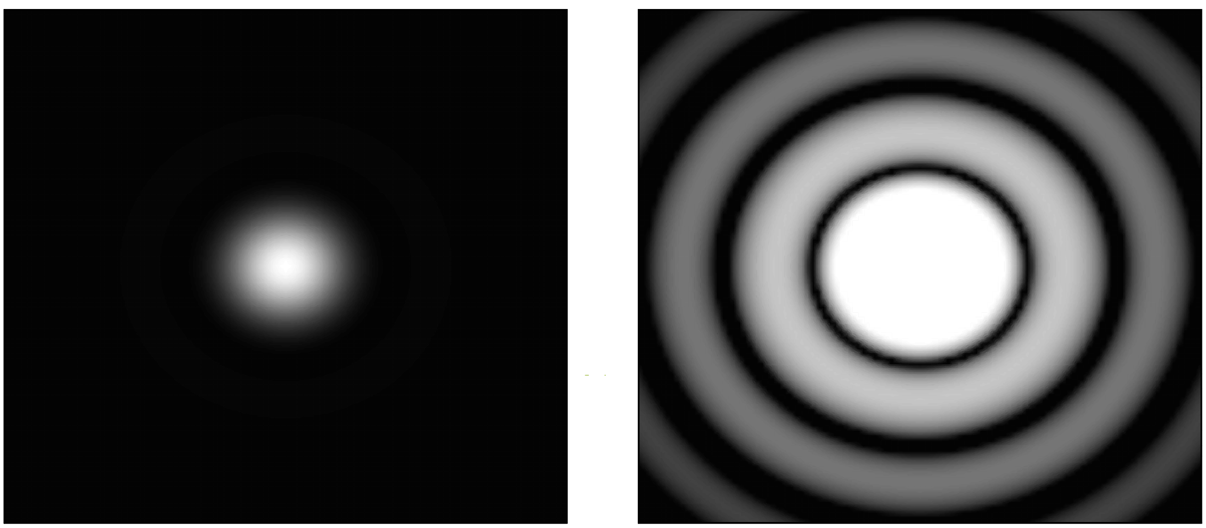

Figure 13.4: grayscale images of the far-field intensity diffracted by a circular aperture. Left: linear scale, for emphasis of the structure of the central maximum. Right: logarithmic scale with the central maximum "saturated" for intensity greater than $10^{-1.3}$ of the peak value, for emphasis of the surrounding bright and dark rings. Figure 13.3 consists of meridional cross sections of these images.

Homework problem 13.1. *Gaussian beams stay Gaussian as they propagate*. Show that a Gaussian near-field distribution with $1/e$ radius ρ_N,

$$E_N(x',y') = E_0 e^{-i\omega t} \exp\left(-\frac{x'^2+y'^2}{\rho_N^2}\right) \; , \tag{13.17}$$

gives rise to a Gaussian far-field distribution,

$$E_F(x,y,z,t) = \frac{\pi\rho_N^2 E_0}{\lambda z} e^{i(\kappa z - \omega t)} \exp\left(-\frac{\pi^2\rho_N^2\left(x^2+y^2\right)}{\lambda^2 z^2}\right) \quad , \tag{13.18}$$

that has $1/e$ radius $\lambda z/\pi\rho_N$.

Homework problem 13.2. Most telescope primary mirrors have central obscurations, in addition to being circular, so their diffraction patterns differ somewhat from Equations 13.14-15.

a. Derive an expression the far-field intensity as a function of $\kappa a\theta$ for an *annular* aperture, with outer half-diameter a and inner half diameter ka ($k < 1$).

b. Plot the intensity divided by peak intensity, $I(\theta,k)/I(0)$, against $\kappa a\theta$, on the same plot with $I(\theta)/I(0)$ for the filled circular aperture, as given by Equations 13.14-15, for k = 0.1 – a rather typical value for telescopes – and the more extreme k = 0.9. What are the major differences between the diffraction patterns of filled circular apertures and annular apertures?