14. Lecture, 19 October 1999

14.1 Diffraction-limited optics: propagation of Gaussian beams

Even simple optical systems take a great deal of effort to design if they are to be used for image detail near the diffraction limit, and if one accounts for diffraction in full generality. Fortunately, there are simple cases – special functional forms of the electric field amplitude distribution in a wavefront – that are fairly close to being spherical waves in the far field, and provide an easy-to-use description of the propagation of light through a system of lenses and mirrors. By far the most useful is the *Gaussian* electric field amplitude function:

$$E_N = E_0 e^{-(\rho'/\rho_N)^2} = E_0 e^{-\left(x'^2+y'^2\right)/\rho_N^2} \quad . \tag{14.1}$$

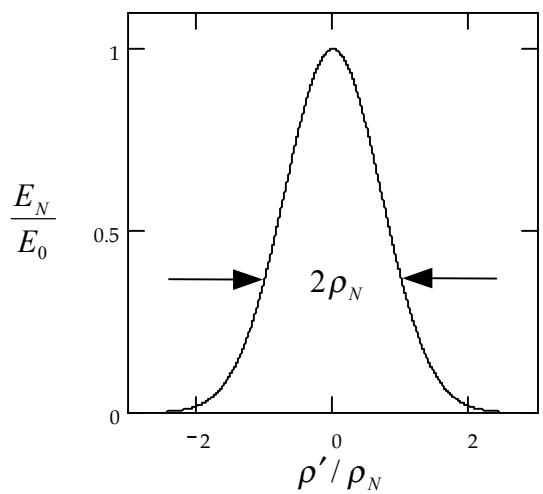

Figure 14.1: Gaussian function.

Here E_0 is a constant, and ρ_N is the cylindrical-radial coordinate value for the E_0/e amplitude level, as shown in Figure 14.1. The nicest property of a Gaussian electric field amplitude is that it *stays* Gaussian as it propagates, because the Fourier transform of a Gaussian is another Gaussian. For instance, as you will show this week in Homework Problem Set 5, a near field with amplitude given by Equation 14.1 leads to the following field a distance $z >> \rho_N$ away:

$$E_F = \frac{\pi\rho_N^2}{\lambda z} E_0 e^{-\kappa^2\rho^2\rho_N^2/4z^2} e^{i(\kappa z - wt)} \quad . \tag{14.2}$$

This is of course just another Gaussian. Its $1/e$ radius is

$$\rho_F = \frac{2z}{\kappa\rho_N} = \frac{\lambda z}{\pi\rho_N} \quad , \tag{14.3}$$

so it's a more spread-out Gaussian, since $z >> \rho_N$. The wavefront is planar at $z = 0$, and is spherical in the far field, just as was the case for the uniformly-illuminated circular aperture; the far-field radius of curvature is z. The whole wavetrain comprises what is called a *Gaussian beam*. The place where the $1/e$

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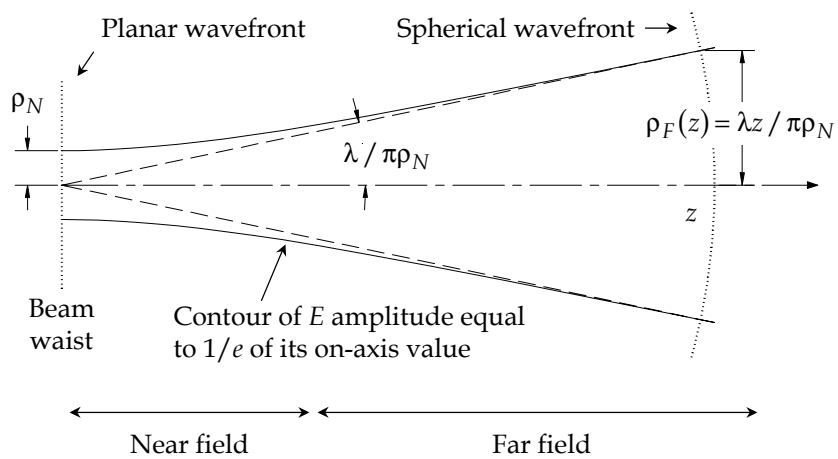

Figure 14.2: Gaussian beam

level of the field amplitude is a local minimum is called a *beam waist*; this occurs only at $z = 0$ in the present example, illustrated in Figure 14.2.

So far we only know expressions for the electric field amplitude at the beam waist and in the far field. For points in between – that is, for z not much larger than ρ_N, a regime indicated in Figure 14.2 as the "near field" – we cannot use the Kirchhoff-integral formulation introduced in Lectures 13 and 14. Treatment of the near-field regime is not particularly difficult, but is lengthy and much more tedious than that of the far-field domain; since we normally work in the far field, we will content ourselves here simply with quoting the results. These results will look plausible enough, given what we know about the far field and the beam waist. The derivations appear in books by D. Marcuse, *Light transmission optics*, chapters 4 and 6, and by J.A. Arnaud, *Beam and fiber optics*, chapter 2. The cylindrical radius to the $1/e$ point in electric field amplitude turns out to be

$$\rho(z) = \left[\rho_N^2 + \left(\frac{\lambda z}{\pi\rho_N}\right)^2\right]^{1/2} \quad . \tag{14.4}$$

Note that $\rho(z)$ tends to ρ_N as $z \to 0$, as it should, and tends to $\lambda z / \pi\rho_N$ for $z >> \rho_N$, also as it should. The radius of curvature of the wavefront is given by

$$R(z) = z\left[1 + \left(\frac{\pi\rho_N^2}{\lambda z}\right)^2\right] \quad , \tag{14.5}$$

which tends to infinity at $z = 0$, as it should (plane wavefront, there), and tends to z for $z >> \rho_N$, also as it should. Note that $R(z)$ has a minimum:

$$\frac{dR}{dz} = 1 - \left(\frac{\pi\rho_N^2}{\lambda}\right)^2 \frac{1}{z^2} = 0$$
$$\text{for} \qquad z_C = \frac{\pi\rho_N^2}{\lambda} \quad . \tag{14.6}$$

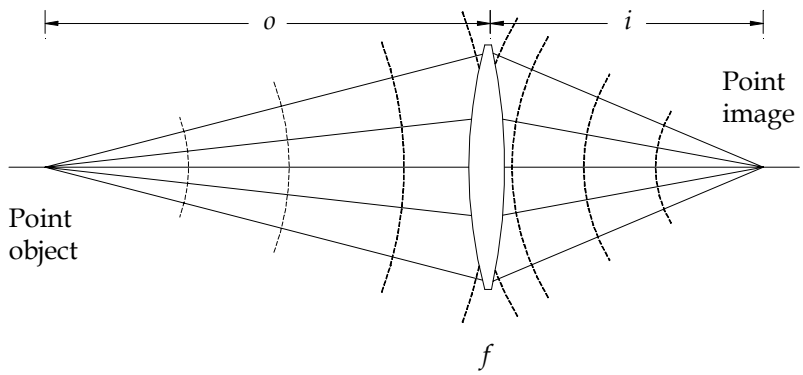

Figure 14.3: paraxial rays and corresponding wavefronts, refocused by a thin lens.

The distance z_C between a beam waist and the following minimum in $R(z)$ is called the *confocal distance*. It provides a convenient division between the near field and the far field – one has to be much more than z_C away from a beam waist in order to justify the use of the far-field approximation.

14.2 Thin lenses and Gaussian beams

Since rays are constructed to be perpendicular to wavefronts, thin lenses, within the confines of paraxial geometrical optics, are devices that simply change the radius of curvature of wavefronts. Consider the point object shown in Figure 14.3; it is a source of spherical waves, which have radius of curvature o when they reach the lens. The waves subsequently converge on the point image, and thus must leave the lens with radius of curvature i. The wavefront *curvatures*, $1/o$ and $1/i$, are of course related by the thin lens equation,

$$\frac{1}{o}+\frac{1}{i}=\frac{1}{f} \quad . \tag{14.7}$$

Since Gaussian-beam wavefronts are spherical, a similar relation holds for *paraxial Gaussian beams*. This useful relation and its consequences are discussed in many texts; the following brief summary will be similar to that by Paul Goldsmith (*Infrared and millimeter waves* **6**, 277-343).

Consider a beam waist with radius ρ_{01} as an object, and a thin lens with focal length f a distance d_1 away along the beam axis, as in Figure 14.4. Suppose that the image forms at a distance d_2 from the lens, with waist radius ρ_{02}. Both beams have spherical wavefronts, and their radii of curvature at the lens are given by Equation 14.5, with $z = d_1$ and d_2 respectively. So the thin lens equation 14.7 gives us

$$\frac{1}{d_1\left[1+\left(\frac{\pi\rho_{01}^2}{\lambda d_1}\right)^2\right]}+\frac{1}{d_2\left[1+\left(\frac{\pi\rho_{02}^2}{\lambda d_2}\right)^2\right]}=\frac{1}{f} \quad . \tag{14.8}$$

Furthermore, the beam radius $\rho(z)$ at the lens must be the same for incoming and outgoing beams, so Equation 14.5 implies

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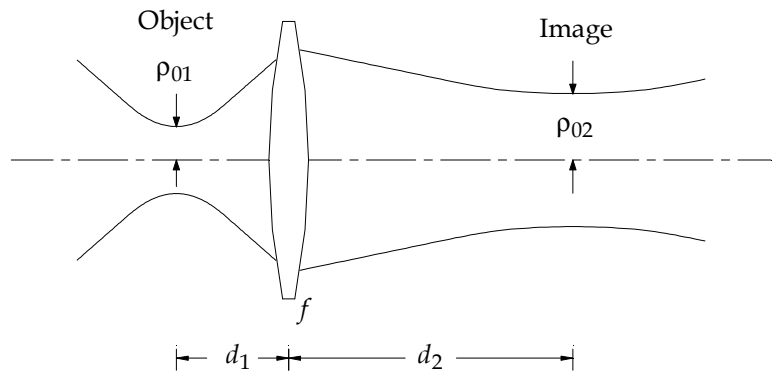

Figure 14.4: paraxial Gaussian beams focussed by a thin lens.

$$d_1\left[1+\left(\frac{\pi\rho_{01}^2}{\lambda d_1}\right)^2\right]=d_2\left[1+\left(\frac{\pi\rho_{02}^2}{\lambda d_2}\right)^2\right] \quad . \tag{14.9}$$

Equations 14.8 and 14.9 can be solved for the image distance and waist size, d_2 and ρ_{02}, in terms of the corresponding parameters for the object, d_1 and ρ_{01}. This is straightforward but not particularly illuminating, so we refer to the interested reader's Mathcad skills, or to Arnaud (*op. cit,* Chapter 2) for the details, and report only the answers:

$$\frac{d_2}{f}=1+\frac{\frac{d_1}{f}-1}{\left(\frac{d_1}{f}-1\right)^2+\left(\frac{\pi\rho_{01}^2}{\lambda f}\right)^2} \qquad \text{Beam waist location}$$

$$\left(\frac{\rho_{02}}{\rho_{01}}\right)^2=\frac{1}{\left(\frac{d_1}{f}-1\right)^2+\left(\frac{\pi\rho_{01}^2}{\lambda f}\right)^2} \qquad \text{Beam waist magnification} \tag{14.10}$$

There is more to these apparently minor modifications to geometrical optics than meets the eye. The expressions differ from paraxial geometrical theory only by the second term in the denominator, but this term contributes important limits – for instance, it keeps the image distance *finite*, essentially preventing thereby the production of collimated beams. There are also ranges for which d_2 increases as d_1 increases, unlike geometrical optics. Finally, image waist locations and magnifications are wavelength-dependent. All of these features are illustrated in Figure 14.5, which is a plot of d_2/f against d_1/f for $\pi\rho_{01}^2/\lambda f$ ranging from zero (the geometrical-optics limit) to unity.

One often desires to transform a given object waist ρ_{01} to a given image waist size ρ_{02}, using a thin lens with given focal length *f*. (This is essentially the one-lens astronomical camera introduced in Lecture 12.) The object and image distances required can be obtained by solving Equations 14.8 and 14.9 simultaneously. Again, skipping the tedious algebra, we get

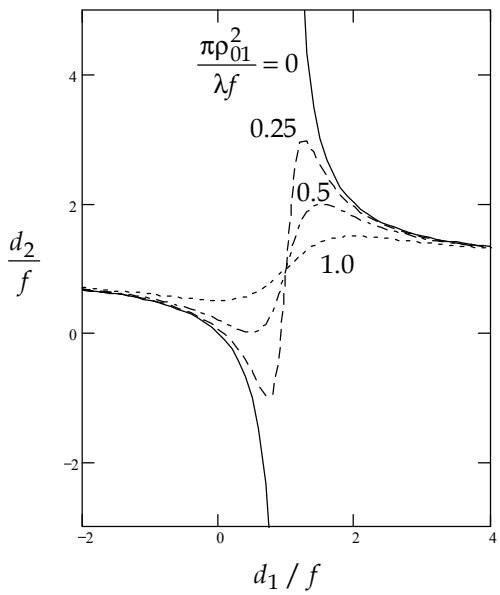

Figure 14.5: image distance *vs.* object distance for a set of beam waist sizes, from Equations 14.10.

$$
\begin{aligned}
d_1 &= f \pm \frac{\rho_{01}}{\rho_{02}}\left[f^2 - \left(\frac{\pi\rho_{01}\rho_{02}}{\lambda}\right)^2\right]^{1/2} \\
d_2 &= f \pm \frac{\rho_{02}}{\rho_{01}}\left[f^2 - \left(\frac{\pi\rho_{01}\rho_{02}}{\lambda}\right)^2\right]^{1/2} \quad ,
\end{aligned}
\tag{14.11}
$$

two solutions, as there must be (recall Example 2.3), one for which the positive sign is taken in both of Equations 14.11 and the other for which the negative sign is taken. It is also assumed that f is greater than $\pi\rho_{01}\rho_{02} / \lambda$, so that the square-root terms are real.

Why are all of these results useful for astronomers? For two reasons, primarily.

1. The central maximum of the Airy pattern for diffraction from a circular aperture, Equation 13.15, accounts for most of the power in the diffraction pattern, and happens to be quite similar to a Gaussian, as is shown in Figure 14.6. Thus Equations 14.8-14.11 provide a good initial approximation for the propagation of "real" diffraction-limited beams through optical systems.

2. In radio astronomy, one often intentionally *tapers* the amplitude response received by the focal-plane instruments so that the effective illumination of the primary mirror is Gaussian rather than uniform. This is done in an effort to supress the diffraction *sidelobes* (features corresponding to the other bright rings of the diffraction pattern), present in the diffraction pattern of a uniformly-illuminated aperture but absent in an ideal Gaussian illumination pattern.

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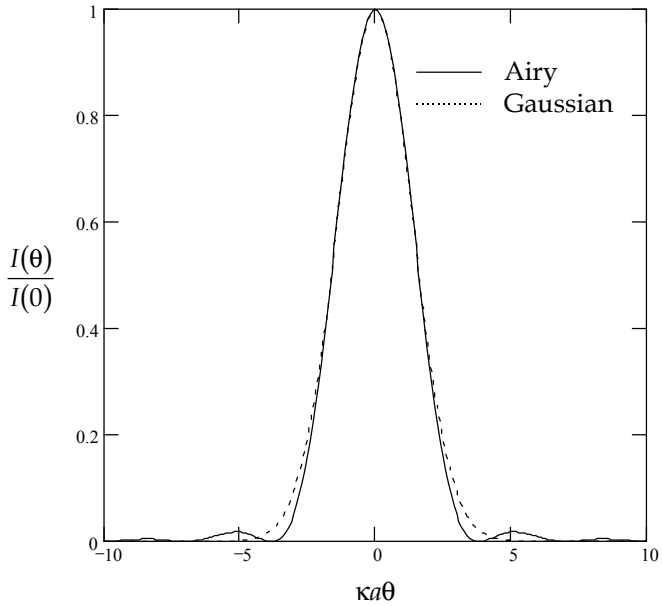

Figure 14.6: Airy pattern and Gaussian beam with the same peak and FWHM diameter, illustrating the similarity of these functions over the range containing the bulk of the power of diffracted light.

Homework problem 14.1. Show that a pair of thin lenses separated by the sum of their focal lengths, with a waist of the input beam at the focal point of one of the lenses, produces an output beam whose waist location is independent of wavelength; then, derive the ratio of the output and input beam waist diameters. This arrangement is usually called a "Gaussian beam telescope," and is useful in radio-astronomy applications in which a large frequency or wavelength range must be used.