26. Lecture, 2 December 1999

26.1 A single vacuum-dielectric interface

Practical optical systems always involve lots of vacuum-dielectric interfaces, so we should use the formalism above to obtain the relevant amplitude transmission and reflection coefficients. This can be done easily with Equations 25.28 and 25.29 by taking medium zero to be vacuum and medium 2 to be a dielectric with index n, so that

$$Y_{0,TE} = \cos\theta_I \qquad Y_{2,TE} = \sqrt{\frac{\varepsilon}{\mu}}\cos\theta_T$$
$$Y_{0,TM} = \frac{1}{\cos\theta_I} \qquad Y_{2,TM} = \sqrt{\frac{\varepsilon}{\mu}}\frac{1}{\cos\theta_T} \tag{26.1}$$

(where $\cos\theta_I$ and $\cos\theta_T$ are related by Snell's law), and by taking medium 1 to have zero thickness, so that the characteristic matrix is simply the unit matrix:

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad . \tag{26.2}$$

Then for TE waves the amplitude transmission and reflection coefficients are

$$t_{TE} = \frac{2Y_0}{Y_0 + Y_2} = \frac{2}{1 + \sqrt{\frac{\varepsilon}{\mu}}\frac{\cos\theta_T}{\cos\theta_I}} \tag{26.3}$$

and

$$r_{TE} = \frac{Y_0 - Y_2}{Y_0 + Y_2} = \frac{1 - \sqrt{\frac{\varepsilon}{\mu}}\frac{\cos\theta_T}{\cos\theta_I}}{1 + \sqrt{\frac{\varepsilon}{\mu}}\frac{\cos\theta_T}{\cos\theta_I}} \quad , \tag{26.4}$$

while for TM waves we get

$$t_{TM} = \frac{2Y_0}{Y_0 + Y_2} = \frac{2}{1 + \sqrt{\frac{\varepsilon}{\mu}}\frac{\cos\theta_I}{\cos\theta_T}} \tag{26.5}$$

and

$$r_{TM} = \frac{Y_0 - Y_2}{Y_0 + Y_2} = \frac{1 - \sqrt{\frac{\varepsilon}{\mu}}\frac{\cos\theta_I}{\cos\theta_T}}{1 + \sqrt{\frac{\varepsilon}{\mu}}\frac{\cos\theta_I}{\cos\theta_T}} \quad . \tag{26.6}$$

For normal incidence on nonmagnetic $(\mu = 1)$ materials these pairs of equations both reduce to

$$t = \frac{2}{1+n} \tag{26.7}$$

and

$$r = \frac{1-n}{1+n} \quad . \tag{26.8}$$

Among other things these expressions imply a limitation to the quantum efficiency of semiconductor detectors when their bare surfaces are illuminated normally, since in this case the quantum efficiency can be no larger than the (power) transmission of the front surface: some of the incident power is reflected. In the argot of optics this is called a *reflection loss*. Most detector semiconductors have rather high index – for instance, n = 3.4 and 4.0 respectively for silicon and germanium, for which the maximum quantum efficiency for bare surfaces is

$$\eta = \frac{Y_2}{Y_0} t^2 = \frac{4n}{(1+n)^2} = 0.70 \text{ and } 0.64 \quad . \tag{26.9}$$

Reflection losses can be reduced or eliminated altogether by using thin dielectric layers as *antireflection coatings*, as you will find in this week's homework (problem 2).

26.2 The Fabry-Perot interferometer

Let us return to the plane-parallel dielectric slab of Figure 25.2, for we are now in a position to calculate the spectrum of its transmission easily. For simplicity we assume that the medium is nonmagnetic and surrounded by vacuum, and that light is incident normally; then

$$Y_{TE} = Y_{TM} = \sqrt{\varepsilon}\cos\theta_T = n \tag{26.10}$$

and

$$\kappa\ell = \frac{2\pi nd}{\lambda}\cos\theta_T = \frac{2\pi nd}{\lambda} \quad , \tag{26.11}$$

so the characteristic matrix of the slab is

$$M = \begin{bmatrix} \cos\kappa\ell & -i\sin\kappa\ell / Y \\ -iY\sin\kappa\ell & \cos\kappa\ell \end{bmatrix} = \begin{bmatrix} \cos\frac{2\pi nd}{\lambda} & -\frac{i}{n}\sin\frac{2\pi nd}{\lambda} \\ -in\sin\frac{2\pi nd}{\lambda} & \cos\frac{2\pi nd}{\lambda} \end{bmatrix} \quad . \tag{26.12}$$

Of course Y = 1 for the media on either side of the slab, so the amplitude transmission coefficient, given by Equation 25.28, is

$$t = \frac{2}{m_{11} + m_{12} + m_{21} + m_{22}} = \frac{2}{2\cos\frac{2\pi nd}{\lambda} - i\left(n + \frac{1}{n}\right)\sin\frac{2\pi nd}{\lambda}} \quad , \tag{26.13}$$

and the (power) transmission is the square modulus of t times the ratio of Ys (1 in this case), or

$$\begin{aligned}\tau = tt^* &= \frac{2}{2\cos\frac{2\pi nd}{\lambda} - i\left(n+\frac{1}{n}\right)\sin\frac{2\pi nd}{\lambda}} \frac{2}{2\cos\frac{2\pi nd}{\lambda} + i\left(n+\frac{1}{n}\right)\sin\frac{2\pi nd}{\lambda}} \\ &= \frac{4}{4\cos^2\frac{2\pi nd}{\lambda} + \left(n+\frac{1}{n}\right)^2 \sin^2\frac{2\pi nd}{\lambda}} = \frac{1}{1+\frac{1}{4}\left[\left(n+\frac{1}{n}\right)^2 - 4\right]\sin^2\frac{2\pi nd}{\lambda}} \\ &= \frac{1}{1+\left[\left(\frac{n^2+1}{2n}\right)^2 - 1\right]\sin^2\frac{2\pi nd}{\lambda}} = \frac{1}{1+\left(\frac{n^2-1}{2n}\sin\frac{2\pi nd}{\lambda}\right)^2} \quad .\end{aligned} \tag{26.14}$$

This function has the expected series of transmission maxima: $\tau = 1$ whenever the sine term vanishes, which takes place whenever the argument of the sine is an integer multiple of π:

$$\frac{2\pi dn\cos\theta_t}{\lambda} = \pi m \qquad (m = 0,1,2,\ldots), \tag{26.15}$$

which is identical to Equation 25.4.

The dielectric slab is more useful as a filter or a spectrometer if the surfaces are more reflective than the bare dielectric we just considered. Note that the bare surface has a power reflectance given from Equation 26.8 by

$$r^2 = \left(\frac{n-1}{n+1}\right)^2 \quad , \tag{26.16}$$

so that

$$1 - r^2 = \frac{(n+1)^2 + (n-1)^2}{(n+1)^2} = \frac{4n}{(n+1)^2} \quad , \tag{26.17}$$

and

$$\frac{2r}{1-r^2} = 2\frac{n-1}{n+1}\frac{(n+1)^2}{4n} = \frac{n^2-1}{2n} \quad . \tag{26.18}$$

We can put this in Equation 26.28 to obtain

$$\tau = \frac{1}{1+\left(\frac{2r}{1-r^2}\sin\frac{2\pi nd}{\lambda}\right)^2} \quad . \tag{26.19}$$

Note that the interference is determined by the phase lags from path lengths through the slab, which do not depend upon the values of reflectivity at the surfaces. Thus Equation 26.19 is more general than Equation 26.14, because the surface reflectivity can be independent of the refractive index: for example, by coating it with a reflective metal or dielectric film. It is often written as

$$\tau = \frac{1}{1 + F \sin^2 \frac{\Delta\phi}{2}} \quad , \tag{26.20}$$

where $\Delta\phi = 4\pi nd / \lambda = 2\kappa nd$, and where

$$F = \left(\frac{2r}{1 - r^2} \right)^2 \tag{26.21}$$

is called the *coefficient of finesse*. The width of the transmission maxima varies with *F*, larger values leading to narrower peaks. The function in Equation 26.20 is called the *Airy function*, again in honor of the former Astronomer Royal of England (and not to be confused with the Airy disk, or the other Airy function, the latter describing the supernumerary arcs of a rainbow). It is plotted in Figure 26.1.

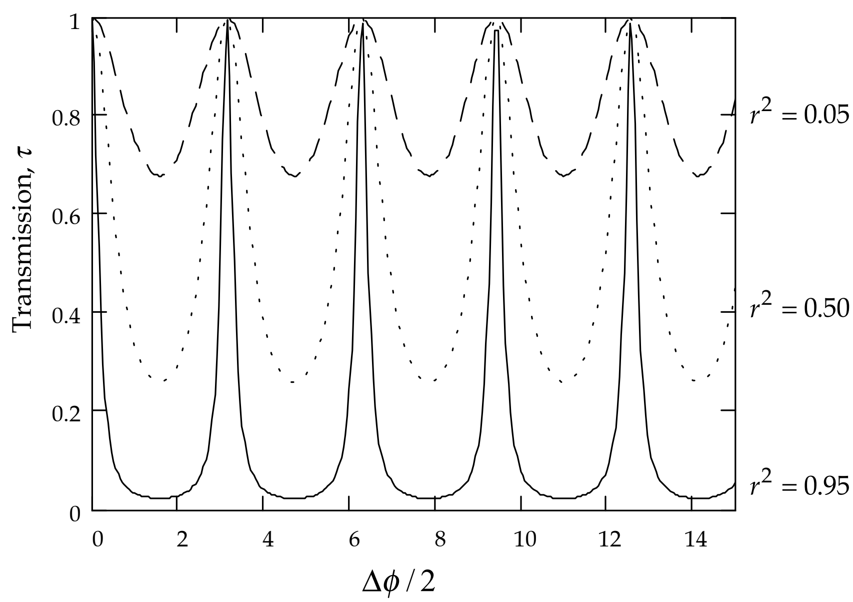

Figure 26.1: the Airy function, the transmission of the Fabry-Perot interferometer.

It is of course possible to perform this derivation for obliquely-incident waves, but not as quickly as this, so we will just give the answer: it turns out that the only change necessary is in the argument of the sine in the denominator of Equation 26.20

$$\Delta\phi = \frac{4\pi nd \cos\theta_T}{\lambda} = 2\kappa nd \cos\theta_T \tag{26.22}$$

(c.f. Equation 25.4).

26.3 Fabry-Perot spectrometers

For large values of the coefficient of finesse, the transmission between peaks is very small, which suggests that "isolating" a single peak is a good way to make a narrow-band (high-resolution) filter. Furthermore, if the reflecting surfaces are placed on *independently movable* substrates, with a non-solid dielectric medium in between, the wavelength at which the peak occurs is tunable. This is the basis of the *Fabry-Perot spectrometer*, a class of astronomical instruments that is good for observation of images of astronomical objects in spectral-line light. Let's compute the resolution of such a filter by calculating the wavelength interval corresponding to the half-peak-transmission points. In terms of $\Delta\phi$, these are found by setting

$$\frac{1}{2} = \frac{1}{1 + F\sin^2\frac{\Delta\phi_{1/2}}{2}} \quad , \tag{26.23}$$

or

$$\sin\frac{\Delta\phi_{1/2}}{2} = \pm\frac{1}{\sqrt{F}} \quad . \tag{26.24}$$

If the peaks are *very* narrow (that is, if F is large), $\Delta\phi$ is very close to $2\pi m$, and the small-angle approximation may be used:

$$\Delta\phi_{1/2} = \pm\frac{2}{\sqrt{F}} \quad . \tag{26.25}$$

The relation between a small interval of $\Delta\phi$ and a small interval of wavelength, $\delta(\Delta\phi)$ and $\delta\lambda$, is thus governed by

$$\begin{aligned} \delta(\Delta\phi) &= \frac{d}{d\lambda}(\Delta\phi)\delta\lambda = \frac{d}{d\lambda}\left(\frac{4\pi nd\cos\theta_T}{\lambda}\right)\delta\lambda = -\frac{4\pi nd\cos\theta_T}{\lambda^2}\delta\lambda \\ &= -\Delta\phi\frac{\delta\lambda}{\lambda} \cong -2\pi m\frac{\delta\lambda}{\lambda} \quad , \end{aligned} \tag{26.26}$$

so

$$\delta\left(\Delta\phi_{1/2}\right) = \pm\frac{2}{\sqrt{F}} = -2\pi m\frac{\delta\lambda_{1/2}}{\lambda} \quad , \tag{26.27}$$

from which we get

$$\Delta\lambda_{FWHM} = \left|\delta\lambda_{1/2.+} - \delta\lambda_{1/2.-}\right| = \frac{2}{\sqrt{F}}\frac{\lambda}{\pi m} \equiv \frac{\lambda}{mQ} \quad . \tag{26.28}$$

Here we have defined the *finesse* of the Fabry-Perot interferometer, $Q = \pi\sqrt{F}/2$. In these terms the (reflectance-limited) spectral resolution of the interferometer is

$$\left(\frac{\Delta\lambda}{\lambda}\right)_{FWHM,r} = \frac{1}{mQ} \quad . \tag{26.29}$$

Note that the spectral resolution gets smaller (improves, in the spectroscopic sense) if the finesse Q or the order number m increases.

The "isolation" of a high-order peak of the Airy function is illustrated in Figure 26.2. If one is interested in observations at a certain wavelength λ_0, say because there is a spectral line there, one can use one Fabry-Perot set in a *low* order there, in series with a higher-order one. A wide-band filter of the type discussed in

§25.1 can be used to reject the higher orders of the low-order Fabry-Perot; this interferometer, if narrow enough, rejects all but one order of the high-order Fabry-Perot. The Fabry-Perot orders can be tuned by adjustment of *d* (or *n*) to scan the detector's response across the spectral line. An example of this concept is explored in extra-credit Problem #7 in this week's homework.

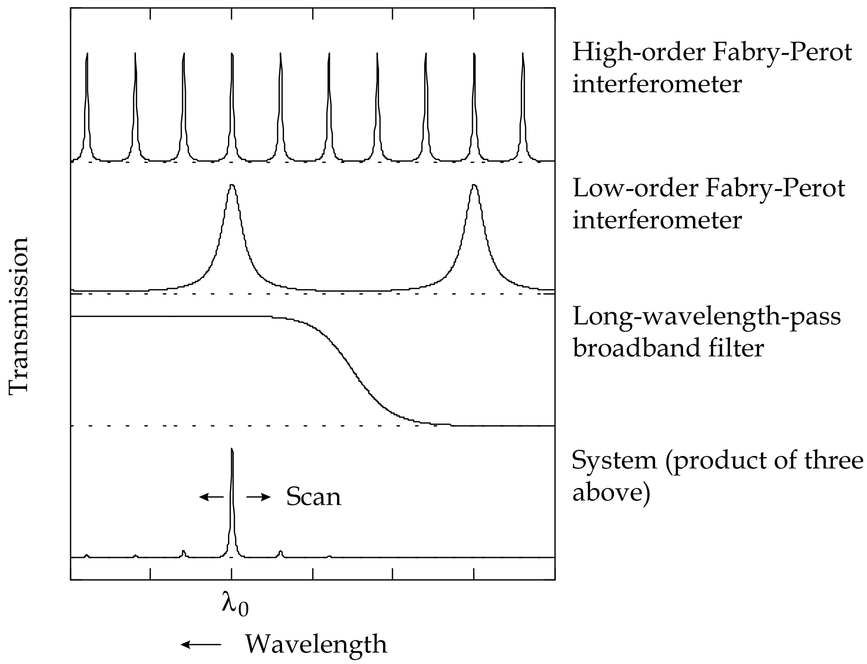

Figure 26.2: Transmission of a tandem Fabry-Perot spectrometer.

Because $\Delta\phi$ depends upon $\cos\theta_T$, reflectance isn't the only thing that can limit the resolution. In Problem 4 of this week's homework, you will show that is a Fabry-Perot is used in a diverging or converging beam of angular radius θ, the resolution can be no better than

$$\left(\frac{\Delta\lambda}{\lambda}\right)_{FWHM,\theta} = \frac{\theta^2}{2} \quad . \tag{26.30}$$

To keep beam divergence or convergence from spreading the beam unnecessarily, we want

$$\left(\frac{\Delta\lambda}{\lambda}\right)_{FWHM,\theta} << \left(\frac{\Delta\lambda}{\lambda}\right)_{FWHM,r} \quad , \tag{26.31}$$

or

$$\frac{\theta^2}{2} << \frac{1}{mQ} \quad . \tag{26.32}$$

For high resolution $(\lambda / \Delta\lambda > 1000)$, this usually means that the high-order Fabry-Perot needs to be illuminated with a *collimated* beam, in order to keep θ as small as possible.

