

# Astronomy 203/203W practice final exam solutions

Fall 1999

No fair peeking at these solutions before attempting to solve the problems yourself!

**1 (50 points)** A 20 cm diameter, 100 cm focal length lens is followed by two smaller lenses, with focal lengths 4.76 cm and 5 cm, as sketched in Figure 1. The first of these lies 104.76 cm past the large lens, and the other lies 9.76 cm further on. All of the lenses are made of flint glass ( $n = 1.5$ ). The arrangement is used as a telescope, with the large lens as the objective (primary) and with a detector in the final focal plane.

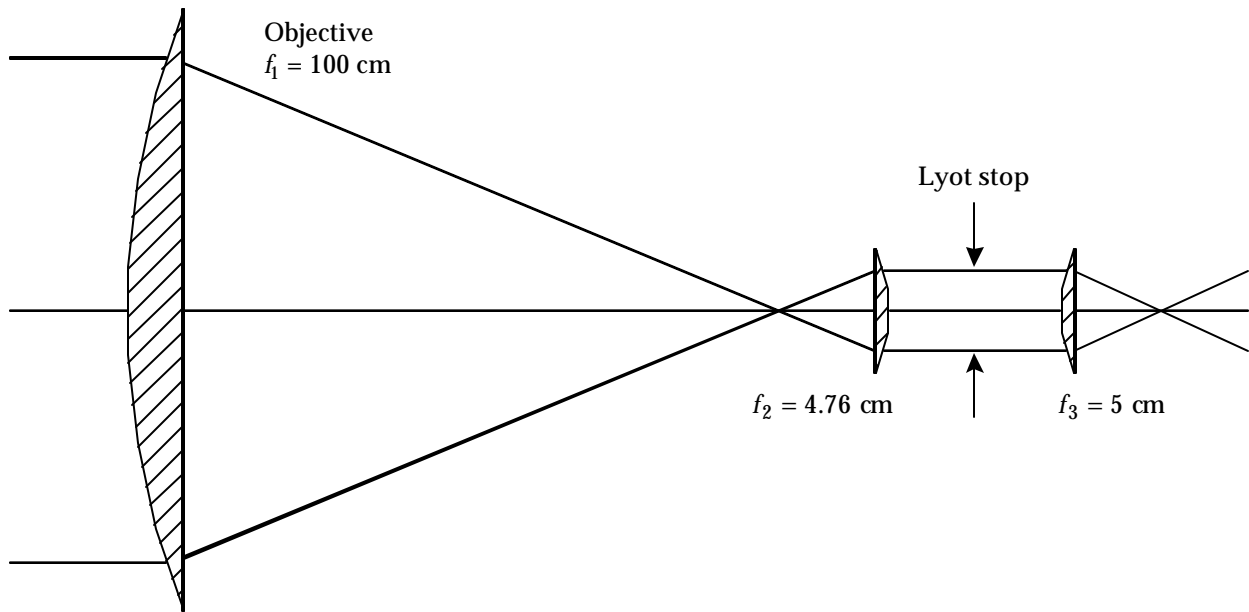


Figure 1

- a. Calculate the plate scale in the focal plane.

$$\begin{aligned}
 PS &= \frac{1}{f_1} \quad \text{at the first focus, and} \\
 &= \frac{1}{f_1} \frac{f_2}{f_3} \quad \text{in the final focal plane;} \\
 &= 9.52 \times 10^{-3} \frac{\text{rad}}{\text{cm}} \left( \frac{180^\circ}{\pi \text{ rad}} \right) \left( \frac{360 \text{ arcsec}}{1^\circ} \right) \left( \frac{\text{cm}}{10 \text{ mm}} \right) \\
 &= 196 \text{ arcsec mm}^{-1} .
 \end{aligned}$$

- b. Calculate the position and diameter of the pupil that would serve best as a Lyot stop; indicate this position on Figure 1.

First find the position and size of the image of the objective formed by the second lens, using  $1/o + 1/i = 1/f$  and  $m = -i/o$ :

$$i_1 = \frac{o_1 f_2}{o_1 - f_2} = 5 \text{ cm} \quad (\text{positive, so it is real and lies to the right of lens 2})$$

$$m_1 = -\frac{i_1}{o_1} = -0.0476 \quad ,$$

$$D_1 = -m_1 D_{\text{objective}} = 0.95 \text{ cm} \quad .$$

This one would work just fine as a Lyot stop, and is marked as such in Figure 1. The next image, for which  $o_2 = 9.76 \text{ cm} - 4.76 \text{ cm} = 4.77 \text{ cm} < f_3$ , is virtual:

$$i_2 = \frac{o_2 f_3}{o_2 - f_3} = -103.7 \text{ cm} \quad ,$$

so it would not be useful as a Lyot stop.

- c. *As drawn, the system suffers from a modest amount of Petzval field curvature. Design a field-flattening lens for use at the focal plane that would eliminate this curvature; calculate an index and focal length for this optical element.*

The Petzval sum is

$$\sum_j \frac{1}{n_j f_j} = \frac{1}{1.5} \left( \frac{1}{100 \text{ cm}} + \frac{1}{4.76 \text{ cm}} + \frac{1}{5 \text{ cm}} \right) = 0.28 \text{ cm}^{-1} \quad ,$$

so the corrector should have

$$\frac{1}{nf} = -\sum_j \frac{1}{n_j f_j} = -0.28 \text{ cm}^{-1} \quad , \text{ or}$$

$$n = 1.5 \quad ,$$

$$f = -2.38 \text{ cm} \quad ,$$

if made from the same glass as the rest of the lenses.

- d. *The second lens is replaced by a lens with focal length -4.76 cm, once again made from glass with refractive index 1.5, and the two smaller lenses, still separated by 9.76 cm, are moved closer to the objective, so that the new negative lens is 95.24 cm from the large lens, as shown in Figure 2. The new system has the same plate scale as the old one, and can in principle perform exactly the same tasks.*

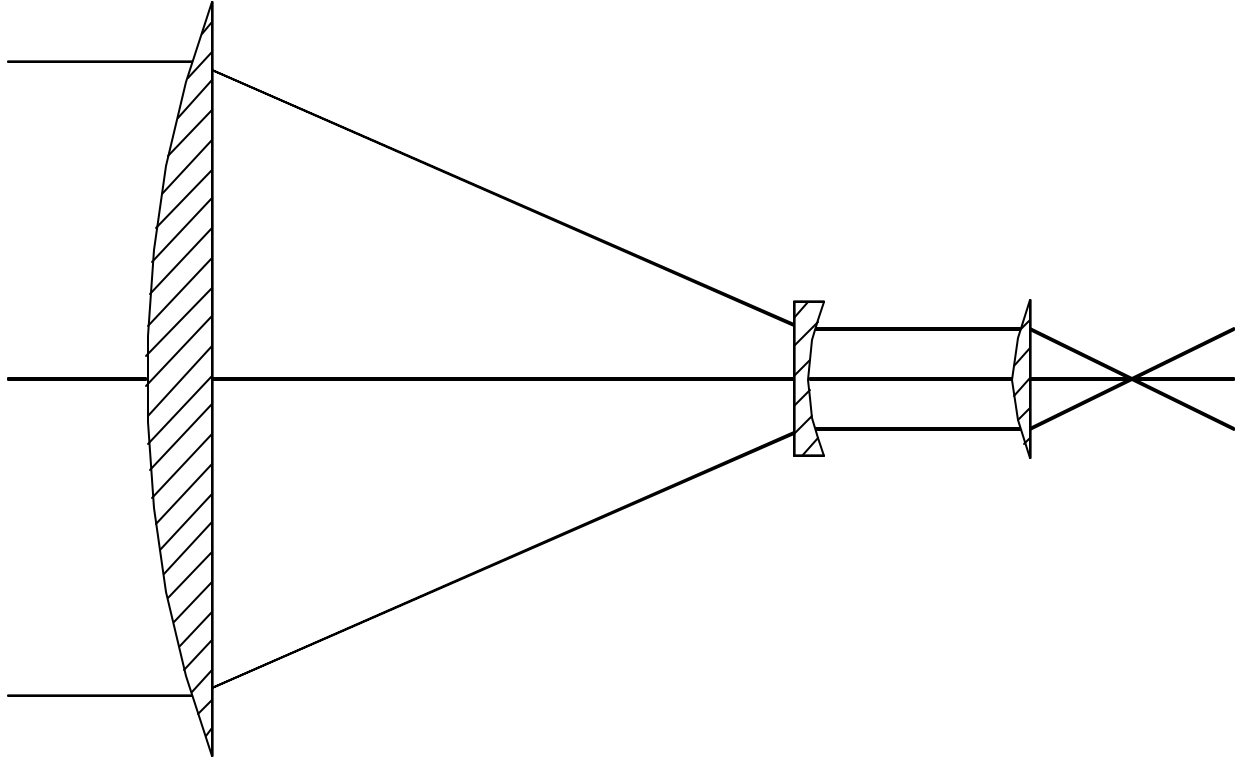


Figure 2

Compared to the previous arrangement, this one has one potential disadvantage and one minor advantage. What are they? (Hint: see parts b and c above)

Disadvantage: no Lyot stop is possible, because the first image of the objective is virtual:

$$i_1 = \frac{o_1 f_2}{o_1 - f_2} = -5.01 \text{ cm} \quad ,$$

and the second image (for which  $o_2 = 14.77 \text{ cm}$ ) lies past the focal plane:

$$i_2 = \frac{o_2 f_3}{o_2 - f_3} = 7.56 \text{ cm} \quad ,$$

Advantage: the Petzval sum is very small:

$$\sum_j \frac{1}{n_j f_j} = \frac{1}{1.5} \left( \frac{1}{100 \text{ cm}} - \frac{1}{4.76 \text{ cm}} + \frac{1}{5 \text{ cm}} \right) = -5.6 \times 10^{-5} \text{ cm}^{-1} \quad ,$$

so no corrector is needed to eliminate field curvature.

**2. (30 points) Coma and astigmatism.**

- a. Consider observations of a 10 arcminute diameter field at the prime focus of a 6 meter focal length paraboloidal telescope. For what telescope diameter will the effects of coma and astigmatism be equal? Which of these is the larger effect for diameters larger and smaller than this?

TTC and TAS are equal when

$$\frac{3}{2} \kappa y^2 \theta = \frac{3}{2} y \theta^2 \quad , \text{ or}$$

$$y = \frac{\theta}{\kappa} = r \theta = 2 f \theta \quad ,$$

where  $\theta$  is the angle of incident rays with respect to the optical axis (5 arcminutes in this case), and  $y$  is the distance from center to edge of the mirror. Thus

$$D = 2y = 4 f \theta = 4 \times 6 \text{ m} \times 5' \times \frac{1^\circ}{60'} \frac{\pi}{180^\circ} = 3.5 \text{ cm} \quad .$$

For larger (smaller) diameters, coma is larger (smaller) than astigmatism. Thus coma is virtually always more important.

- b. Now consider observations at the prime focus of a 2 meter diameter, 6 meter focal length paraboloidal telescope. For what angular diameter will the effects of coma and astigmatism be equal? Which of these is the larger effect for angular diameter larger and smaller than this?

Use  $y = 2 f \theta$  again:

$$\theta = \frac{y}{2 f} = \frac{D}{4 f} = 0.083 \text{ rad} = 4.8^\circ \quad .$$

At larger angles than this, astigmatism is bigger; at smaller angles, coma. 4.8 degrees is a very large field of view for a 2 m telescope; again this means that coma is more important than astigmatism in most cases.

**3. (30 points)** The spot diagram in Figure 3 was produced in the paraxial focal plane of a spherical mirror 20 cm in diameter. The incident rays were arranged in concentric circles, and spread all the way to the edge of the mirror (like RayTrace's "Bullseye" pattern).

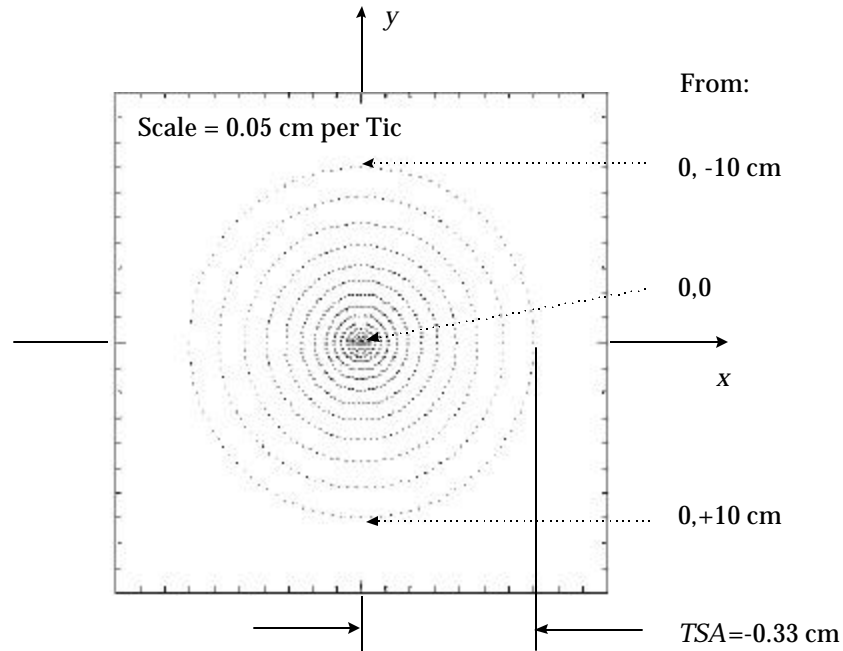


Figure 3

- a. On Figure 3, identify the spots corresponding to the rays which reflected from the mirror at coordinates  $(x, y) = (0, 10 \text{ cm}), (0, 0),$  and  $(0, -10 \text{ cm}).$

See Figure 3.

- b. What is this mirror's radius of curvature?

Measure it on Figure 3. I get  $TSA = -0.33 \text{ cm}$ , which for  $\epsilon = 0$  and to third order this is equal to

$$TSA = -0.33 \text{ cm} = -\frac{1}{2} \kappa^2 y^3, \text{ so}$$

$$r = \frac{1}{\kappa} = \sqrt{-\frac{y^3}{2TSA}} = 39 \text{ cm} .$$

**4. (50 points)** A Gregorian telescope, depicted in Figure 4, has a primary mirror with diameter 2 m and apex radius of curvature 6 m. Its secondary mirror has eccentricity 0.5 and apex radius of curvature 3 m.

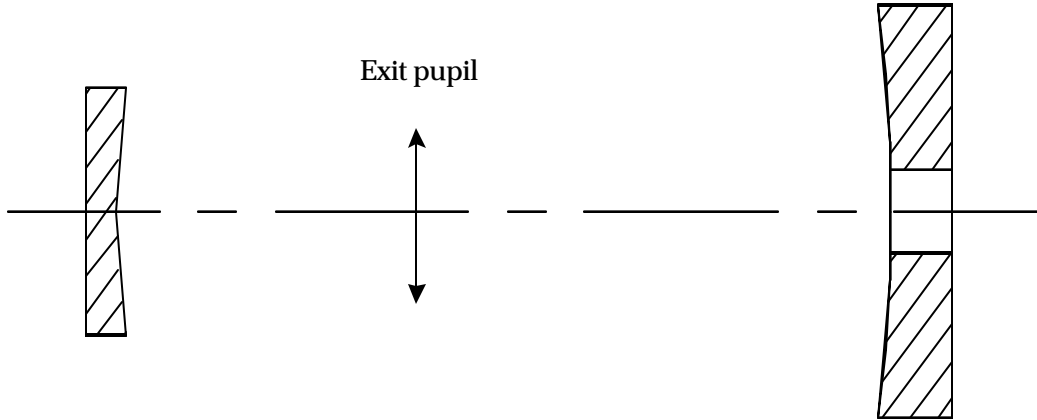


Figure 4

- a. Calculate the distances from the secondary mirror's apex to the primary's apex, and from the secondary's apex to the focal plane.

The secondary is an ellipsoid; call its short and long focal lengths  $f_1$  and  $f_2$ , respectively. Its radius of curvature and eccentricity are given by

$$r = \frac{b^2}{a}$$

$$\epsilon = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{a^2 - ar}}{a} ,$$

whence

$$a^2 \epsilon^2 = a^2 - ar$$

$$a = \frac{r}{1 - \epsilon^2} = \frac{3 \text{ m}}{1 - \frac{1}{4}} = 4 \text{ m} \Rightarrow c = ae = 2 \text{ m} ,$$

and

$$f_1 = c - a = 2 \text{ m} ,$$

$$f_2 = c + a = 6 \text{ m} .$$

The primary's focus lies  $f_p = r_p / 2 = 3 \text{ m}$  from its apex, and must be coincident with the near focus of the secondary, so the apex separation is  $f_1 + f_p = 5 \text{ m}$ , and the focal plane is  $f_2 = 6 \text{ m}$  from the secondary's apex, or 1 m behind the primary's.

- b. Calculate the plate scale in the final, Gregorian, focal plane, in arcsec  $\text{mm}^{-1}$ .

$$PS = \frac{1}{f_p} \frac{f_1}{f_2} = \frac{1}{9} \text{ rad m}^{-1} = 22.9 \text{ arcsec mm}^{-1} .$$

- c. Consider the primary's edge to be the aperture stop, and calculate the diameter of the telescope's exit pupil, and the distance from this pupil to the secondary's apex.

The effective focal length of the secondary is

$$f = \frac{f_1 f_2}{f_1 + f_2} = 1.5 \text{ m} .$$

$o = 5 \text{ m}$  (from part a), so

$$i = \frac{of}{o - f} = +2.14 \text{ m} ,$$

$$D_{EP} = -\frac{i}{o} D_p = 85.7 \text{ cm} .$$

- d. Indicate the position and size of the exit pupil on the telescope diagram shown in Figure 4.

Done.

**5. (30 points)** A small, concave paraboloidal mirror, with diameter 2 cm and apex radius of curvature 200 cm, is used to make an image of two stars which are separated by an angle of 0.01 radian in the sky.

- a. First, the mirror is aligned so that the light from the two stars is incident on the mirror at equal and opposite angles from the paraboloid's axis. Compute the distance between the stellar images in the focal plane.

$$d = f\theta = r\theta / 2 = 1 \text{ cm} .$$

- b. Next, the mirror is tilted so that one star's light is incident at an angle 0.1 radian from the paraboloid's axis, and the other star's light at 0.11 radian from this axis. The stellar images in the focal plane remain sharp, but appear to lie further apart than was the case in part a. Which aberration gives rise to this increase in the separation of the images? By how much did the distance between the images increase?

The plate scale increases with distance from the center of the focal plane: this is the defining characteristic of pincushion distortion. The third-order expression for transverse distortion is

$$TD = -\frac{3\theta^3}{4\kappa} ,$$

so each image lies  $3\theta^3 / 4\kappa$  further away from the center than the paraxial plate scale would have it, and the separation of the images is greater by

$$\Delta y = \frac{3}{4\kappa} (\theta_2^3 - \theta_1^3) = 0.05 \text{ cm} .$$

**6. (50 points) Reflection losses and detector quantum efficiency.**

- a. Calculate the transmission of the planar surface of a dielectric medium with index of refraction  $n$  for light of wavelength  $\lambda$  incident normally from vacuum.

For normal incidence  $TE$  waves are the same as  $TM$ , so

$$Y_0 = 1 \quad Y_2 = \sqrt{\frac{\epsilon}{\mu}} = n \quad \text{if as usual } \mu = 1.$$

If there is no medium separating the vacuum (medium 0) and the dielectric (medium 2), then the appropriate characteristic matrix is the unit matrix. Thus

$$t = \frac{2Y_0}{m_{11}Y_0 + m_{12}Y_0Y_{p+1} + m_{21} + m_{22}Y_{p+1}} = \frac{2}{1+n},$$

and

$$\tau = \frac{Y_{p+1}}{Y_I} |t|^2 = \frac{4n}{(1+n)^2}.$$

- b. Repeat the calculation for the same surface coated with a plane-parallel dielectric film with index  $n' = \sqrt{n}$  and thickness  $\lambda / 4n'$ .

Now there is a medium 1, for which

$$Y_1 = \sqrt{n},$$

$$\kappa\ell = \frac{2\pi\sqrt{n}}{\lambda} \frac{\lambda}{4\sqrt{n}} = \frac{\pi}{2}, \quad \text{and}$$

$$M_1 = \begin{bmatrix} \cos \kappa\ell & -i \sin \kappa\ell / Y_1 \\ -i Y_1 \sin \kappa\ell & \cos \kappa\ell \end{bmatrix} = \begin{bmatrix} 0 & -i/\sqrt{n} \\ -i\sqrt{n} & 0 \end{bmatrix},$$

so that

$$t = \frac{2Y_0}{m_{11}Y_0 + m_{12}Y_0Y_{p+1} + m_{21} + m_{22}Y_{p+1}} = \frac{2}{\frac{-i}{\sqrt{n}}n - i\sqrt{n}} = -\frac{1}{i\sqrt{n}},$$

and

$$\tau = \frac{Y_{p+1}}{Y_I} |t|^2 = n \frac{1}{n} = 1. \quad (!)$$

**7. (70 points) Extragalactic observations of the [O III] 0.5007 mm line.** A certain spiral galaxy, at rest with respect to the Sun and seen nearly edge-on, emits the [O III] 0.5007 mm line throughout its disk. The galaxy rotates with circular speed  $360 \text{ km s}^{-1}$ , so that on one side of the galactic nucleus the line appears at  $\lambda = 0.5013 \text{ mm}$ , and on the other side it appears at  $\lambda = 0.5001 \text{ mm}$ .



- a. Consider observations of this galaxy with a normal-incidence grating spectrometer, in first order. The groove spacing of the grating is  $a = 2.0 \text{ mm}$ . At what angles (in degrees) will the wavelengths corresponding to the limits of galactic rotation be dispersed?

$$a(\sin \theta_m - \sin \theta_i) = m\lambda \quad .$$

Here  $a = 2 \text{ } \mu\text{m}$  and  $\theta_i = 0$ , so

$$\begin{aligned} \theta_m &= \sin^{-1} \frac{\lambda}{a} \\ &= 14.480^\circ \quad \text{for } \lambda = 0.5001 \text{ } \mu\text{m} \text{ ,} \\ &= 14.516^\circ \quad \text{for } \lambda = 0.5013 \text{ } \mu\text{m} \text{ .} \end{aligned}$$

- b. At what angle (in degrees) should the grating be blazed, to ensure optimum efficiency?

Blaze it for the middle of this angular range; that is, at  $\lambda = 0.5007 \text{ } \mu\text{m}$  :

$$\begin{aligned} \theta_{m0} &= \sin^{-1} \frac{\lambda_0}{a} = 14.497^\circ \text{ ;} \\ \gamma &= \frac{\theta_{m0} + \theta_i}{2} = 7.25^\circ \text{ .} \end{aligned}$$

- c. A camera lens is used to focus the dispersed light onto a CCD detector array with pixels spaced by  $10 \text{ mm}$ . It is desired that the wavelength range of galactic rotation correspond to 20 detector spacings. What should the focal length of the camera lens be?

The range of angles is  $\Delta\theta = 14.516^\circ - 14.480^\circ = 0.036^\circ = 6.2 \times 10^{-4} \text{ rad}$ . The plate scale of the lens should be  $\Delta\theta / 20$  divided by the pixel spacing:

$$\frac{1}{f} = \frac{\Delta\theta}{20\Delta x} \Rightarrow f = 32.3 \text{ cm} \quad .$$

- d. It is also desired that the wavelength range of galactic rotation correspond to 10 spectral resolution elements (FWHM). How many grating rulings need to be illuminated, and what is the diameter of the beam incident on the grating?

We'll assume again that the middle of the range ( $\lambda = 0.5007 \text{ } \mu\text{m}$ ) is meant:

$$\begin{aligned} \delta\lambda &= 0.5013 \text{ } \mu\text{m} - 0.5001 \text{ } \mu\text{m} = 0.0012 \text{ } \mu\text{m} = 10\Delta\lambda_{\text{FWHM}} \\ \frac{\Delta\lambda_{\text{FWHM}}}{\lambda_0} &= \frac{1}{mN} = \frac{1}{N} \\ N &= \frac{\lambda_0}{\Delta\lambda_{\text{FWHM}}} = \frac{10\lambda_0}{\delta\lambda} \cong 4200 \text{ .} \end{aligned}$$

The grating rulings are  $a = 2 \text{ } \mu\text{m}$  apart, so the illuminated length of the grating (that is, the beam diameter) is  $L = Na = 0.84 \text{ cm}$  .

- e. At  $\lambda = 0.5007 \text{ nm}$ , good Fabry-Perot interferometers have finesse typically equal to  $Q = 30$ . In what order would such a Fabry-Perot need to be operated if it were to have the same FWHM spectral resolution as the grating spectrometer discussed above?

$$\begin{aligned} \frac{\Delta\lambda}{\lambda_0} &= \frac{1}{mQ} \quad (\text{Fabry - Perot}) \\ &= \frac{1}{N} \quad (\text{this grating}) \end{aligned} \Rightarrow m = \frac{N}{Q} \cong 140 \text{ .}$$

- f. By how much would the Fabry-Perot mirror spacing need to be changed for this order to be scanned over the wavelength range of galactic rotation?

Assume normal incidence and  $n = 1$  between the mirrors:

$$\begin{aligned} \lambda_m &= \frac{2d}{m} \Rightarrow d = \frac{m\lambda_m}{2} = 34.757 \text{ } \mu\text{m} - 34.840 \text{ } \mu\text{m} \text{ for the full range;} \\ \Delta d &= 0.083 \text{ } \mu\text{m} \text{ .} \end{aligned}$$

### 8. (70 points) Photon noise and Johnson noise.

- a. Starting with the Bose-Einstein probability distribution for photons, show that the standard deviation in photon number per mode in blackbody radiation is given by

$$\Delta N = \sqrt{\bar{N}(\bar{N} + 1)} \text{ ,}$$

where, as usual,

$$\bar{N} = \frac{1}{e^{h\nu/kT} - 1} \text{ .}$$

Hint: recall that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ .}$$

That probability distribution is of course

$$p(N)dN = (1 - e^{-h\nu/kT})e^{-Nh\nu/kT} \text{ .}$$

As usual, let  $x = h\nu / kT$ , and calculate the variance of  $N$ :

$$\begin{aligned}
\overline{N^2} &= \sum_{N=0}^{\infty} N^2 p(N) = (1 - e^{-x}) \sum_{N=0}^{\infty} N^2 e^{-Nx} = (1 - e^{-x}) \sum_{N=0}^{\infty} \frac{\partial^2}{\partial x^2} e^{-Nx} = (1 - e^{-x}) \frac{d^2}{dx^2} \sum_{N=0}^{\infty} (e^{-x})^N \\
&= (1 - e^{-x}) \frac{d^2}{dx^2} \frac{1}{1 - e^{-x}} = -(1 - e^{-x}) \frac{d}{dx} \frac{e^{-x}}{(1 - e^{-x})^2} = (1 - e^{-x}) \left[ \frac{e^{-x}}{(1 - e^{-x})^2} + 2e^{-x} \frac{e^{-x}}{(1 - e^{-x})^3} \right] \\
&= \frac{e^{-x} - e^{-2x} + 2e^{-2x}}{(1 - e^{-x})^2} = \frac{e^{-x} + e^{-2x}}{(1 - e^{-x})^2} \left( \frac{e^{2x}}{e^{2x}} \right) = \frac{e^x + 1}{(e^x - 1)^2} = \overline{N}^2 \left( 1 + \frac{1}{\overline{N}} + 1 \right) = 2\overline{N}^2 + \overline{N} \quad ,
\end{aligned}$$

and then the standard deviation:

$$\Delta N = \sqrt{\overline{N^2} - \overline{N}^2} = \sqrt{2\overline{N}^2 + \overline{N} - \overline{N}^2} = \sqrt{\overline{N}(\overline{N} + 1)} \quad . \quad (\text{Q.E.D.})$$

- b. *A photodetector with resistance  $R$  absorbs a single mode and polarization of radiation emitted from a blackbody. The detector is hooked up in a circuit in such a way that there is no DC photocurrent or shot noise present, but only AC currents like those from the photon-number fluctuations calculated in part a. Take the electrical noise power per unit bandwidth dissipated in the detector to be equal to the power fluctuations corresponding to these photon-noise fluctuations, and derive an expression for the standard deviation of current in the detector. Show that in the Rayleigh-Jeans limit ( $h\nu \ll kT$ ) this expression reduces to the familiar expression for Johnson noise,*

$$\Delta I = \sqrt{\frac{kT\Delta\nu}{R}} \quad .$$

The blackbody power can be written for a single spatial mode ( $A\Omega = \lambda^2$ ) and polarization as

$$P_B = \overline{P}_B = \frac{1}{2} B_\nu(T) \Delta\nu A\Omega = h\nu\Delta\nu\overline{N} \quad .$$

Thus

$$\overline{P}_B^2 = (h\nu\Delta\nu)^2 \overline{N}^2 \quad ,$$

and

$$\begin{aligned}
\Delta P_B &= h\nu\Delta\nu\Delta N = h\nu\Delta\nu\sqrt{\overline{N}(\overline{N} + 1)} \\
&= \overline{N}h\nu\Delta\nu = kT\Delta\nu \quad \text{in the Rayleigh - Jeans limit} \quad .
\end{aligned}$$

Set this power equal to that dissipated by the current  $\Delta I$ :

$$\begin{aligned}
(\Delta I)^2 R &= kT\Delta\nu \\
\Delta I &= \sqrt{\frac{kT\Delta\nu}{R}} \quad .
\end{aligned}$$

**9. (70 points)** A Cassegrain telescope, shown in Figure 5, is followed by a pair of lenses set up as collimator and camera. Its parameters are: primary diameter 1 m, primary focal length 2 m, secondary diameter 0.25 m, secondary focal lengths 0.5 m, 2.5 m, collimator focal length 0.2 m, camera focal length 0.1 m, collimator-camera lens separation 30 cm. It is used for observations at a wavelength of 100  $\mu\text{m}$ .

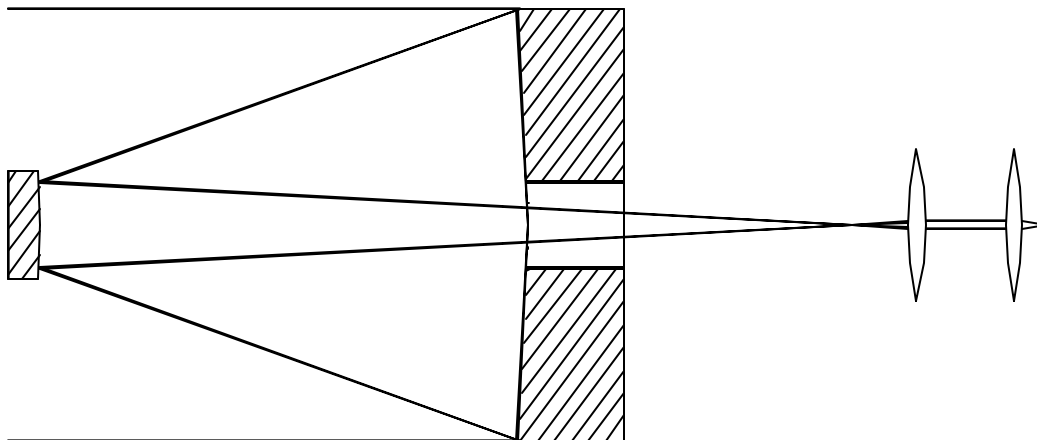


Figure 5

- a. What is the FWHM diameter of the diffraction spot at the Cassegrain focus?

$$\text{Plate scale: } \frac{1}{f_{\text{eff}}} = \frac{1}{f_p} \frac{f_1}{f_2} .$$

$$\Delta\theta_{FWHM} = 1.2 \frac{\lambda}{D} \Rightarrow d_{FWHM} = f_{\text{eff}} \Delta\theta_{FWHM} = 1.2 \frac{f_p f_2}{f_1} \frac{\lambda}{D} = 1.2 \text{ mm} .$$

- b. A detector lies at the focus of the camera lens, and is exactly the size of the FWHM diffraction spot at this focus. What is its diameter?

Since the lenses are spaced by the sum of their focal lengths, they comprise a Gaussian-beam telescope (c.f. Problem Set #5, problem 1), so the magnification is  $f_2 / f_1$  whether paraxial or Gaussian-beam optics is used:

$$d'_{FWHM} = \frac{f_2}{f_1} d_{FWHM} = 0.6 \text{ mm} .$$

- c. There is a place between the collimator and camera lenses at which the diffraction-limited beam diameter reaches a minimum. Where is it, and what is the FWHM diameter of the beam there?

First,

$$\frac{d_2}{f_1} = 1 + \frac{\frac{d_1}{f_1} - 1}{\left(\frac{d_1}{f_1} - 1\right)^2 + \left(\frac{\pi \rho_{01}^2}{\lambda f_1}\right)^2},$$

but  $d_1 = f_1$ , so

$$d_2 = f_1 ;$$

the beam waist lies  $f_1 = 20$  cm past the collimator lens. For the waist radii,

$$\left(\frac{\rho_{02}}{\rho_{01}}\right)^2 = \frac{1}{\left(\frac{d_1}{f_1} - 1\right)^2 + \left(\frac{\pi \rho_{01}^2}{\lambda f_1}\right)^2} = \left(\frac{\lambda f_1}{\pi \rho_{01}^2}\right)^2, \text{ or}$$

$$\frac{\rho_{02}}{\rho_{01}} = \frac{\lambda f_1}{\pi \rho_{01}^2}.$$

Take  $\rho_{01}$  to be the radius of the Cassegrain focus diffraction spot (about 0.6 mm), so

$$2\rho_{02} = 2.1 \text{ cm}.$$

- d. A cryogenic filter with  $DI/l = 10^{-2}$  and transmission 0.2 is placed in front of the detector, which is also held at a cryogenic temperature. The telescope's temperature is 300 K, and its effective emissivity is 0.1. The detector's quantum efficiency is 0.5, and its photoconductive gain and gain dispersion are both 1. What is the background-limited noise equivalent power (NEP) of the system?

The detector is the same size as the FWHM diffraction spot, so  $A\Omega = \lambda^2$ , and

$$P_B = \varepsilon B_\nu(T) \Delta\nu A\Omega = 2\varepsilon h\nu \bar{N} \frac{\nu}{100},$$

where

$$\bar{N} = \frac{1}{e^{h\nu/kT} - 1} = 1.63$$

for  $T = 300$  K and  $\lambda = 100$   $\mu\text{m}$ . Thus

$$NEP = \frac{1}{1 - \varepsilon} \sqrt{\frac{\beta}{\tau\eta} 2h\nu \cdot 2\varepsilon h\nu \bar{N} \frac{\nu}{100} \left(1 + \frac{\tau\eta\varepsilon}{\beta} \bar{N}\right)} = 9.8 \times 10^{-16} \text{ W Hz}^{-1/2}.$$

- e. A certain compact celestial object produces a flux of  $1.3 \times 10^{-15} \text{ W m}^{-2}$  at the Earth's surface, within the bandwidth of the filter used in part d. How long would it take the system to detect this object with a signal-to-noise ratio of 10?

$$P_S = \text{flux} \times A_{\text{telescope}} = \text{flux} \times \pi y_1^2 = 10^{-15} \text{ W}$$

$$\frac{S}{N} = \frac{P_S}{NEP} \sqrt{2 \Delta t}$$

$$\Delta t = \frac{1}{2} \left( \frac{S}{N} \frac{NEP}{P_S} \right)^2 = 46 \text{ seconds.}$$

**10. (50 points)** Calculate the intensity on a screen in the far field, as a function of angle with respect to the optical axis, of light diffracted by a square aperture with side  $a$ .

$$\begin{aligned} E_F(\kappa_x, \kappa_y, t) &= \frac{e^{i\kappa r}}{\lambda r} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_N(x', y', t) e^{-i(\kappa_x x' + \kappa_y y')} dx' dy' = \frac{e^{i(\kappa r - \omega t)} E_0}{\lambda r} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} e^{-i(\kappa_x x' + \kappa_y y')} dx' dy' \\ &= \frac{e^{i(\kappa r - \omega t)} E_0}{\lambda r} \left. \frac{e^{-i\kappa_x x'}}{-i\kappa_x} \right|_{-a/2}^{a/2} \left. \frac{e^{-i\kappa_y y'}}{-i\kappa_y} \right|_{-a/2}^{a/2} = \frac{4e^{i(\kappa r - \omega t)} E_0}{\lambda r \kappa_x \kappa_y} \sin\left(\frac{\kappa_x a}{2}\right) \sin\left(\frac{\kappa_y a}{2}\right) \end{aligned}$$

$$\begin{aligned} I(\kappa_x, \kappa_y) &= \frac{c}{8\pi} E_F E_F^* \quad (\text{CGS units}) \\ &= \left( \frac{4E_0}{\lambda r \kappa_x \kappa_y} \right)^2 \sin^2\left(\frac{\kappa_x a}{2}\right) \sin^2\left(\frac{\kappa_y a}{2}\right) \end{aligned}$$

Recall that  $\kappa_x = \kappa \theta_x = \kappa x / r$ , and similarly for  $\kappa_y$  where  $x$  and  $y$  are rectangular coordinates on the screen; then

$$I(\theta_x, \theta_y) = \left( \frac{E_0 a}{\lambda r} \right)^2 \left( \frac{\sin\left(\frac{\kappa a \theta_x}{2}\right)}{\frac{\kappa a \theta_x}{2}} \right)^2 \left( \frac{\sin\left(\frac{\kappa a \theta_y}{2}\right)}{\frac{\kappa a \theta_y}{2}} \right)^2.$$

Furthermore, note that

$$\lim_{\theta \rightarrow 0} \frac{\sin\left(\frac{\kappa a \theta}{2}\right)}{\frac{\kappa a \theta}{2}} = \lim_{\theta \rightarrow 0} \frac{\cos\left(\frac{\kappa a \theta}{2}\right)}{\frac{\kappa a}{2}} = \frac{2}{\kappa a}$$

by l'Hôpital's rule, so

$$I_0 = I(0,0) = \left( \frac{E_0 a}{\lambda r} \right)^2 \left( \frac{2}{\kappa a} \right)^4,$$

and

$$I(\theta_x, \theta_y) = I_0 \left( \frac{\kappa a}{2} \right)^4 \left( \frac{\sin\left(\frac{\kappa a \theta_x}{2}\right)}{\frac{\kappa a \theta_x}{2}} \right)^2 \left( \frac{\sin\left(\frac{\kappa a \theta_y}{2}\right)}{\frac{\kappa a \theta_y}{2}} \right)^2 .$$

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