

# Physics 217 Makeup Midterm Exam

29 October 2002

If any of these answers seems obscure, please ask us questions about it until we clear it up for you.

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## Problem 1 (20 points)

- a. Evaluate  $\nabla \cdot (r^n \hat{r})$ , for all integer values of  $n$ . Pay close attention to  $n = -2$ .

$$\begin{aligned}\nabla \cdot (r^n \hat{r}) &= \frac{1}{r^2} \frac{d}{dr} (r^2 r^n) = \frac{(n+2)r^{n+1}}{r^2} \quad (\text{except for } n = -2) \\ &= \boxed{(n+2)r^{n-1}} \quad , n \neq -2.\end{aligned}$$

The  $n = -2$  case is undefined: officially the derivative comes out to be zero, but on the other hand the direction is not defined for vectors of zero length. We have seen that this leads to problems with satisfaction of the divergence theorem (part b) unless

$$\nabla \cdot (r^{-2} \hat{r}) = \boxed{4\pi \delta^3(\mathbf{r})} .$$

- b. Verify that your solutions for part a are correct by using the divergence theorem (that is, calculate  $\int (r^n \hat{r}) \cdot d\mathbf{a}$  and  $\int \nabla \cdot (r^n \hat{r}) d\tau$ ; do they differ?).

$$\begin{aligned}\oint r^n \hat{r} \cdot d\mathbf{a} &= \int r^n r^2 \sin\theta d\theta d\phi = r^{n+2} \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \\ &= r^{n+2} (2)(2\pi) = \boxed{4\pi r^{n+2}} \quad (n \neq 2), \\ \int \nabla \cdot (r^n \hat{r}) d\tau &= \int (n+2) r^{n-1} r^2 dr d\theta d\phi \\ &= 4\pi (n+2) \int_0^r r'^{n+1} dr' \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi' \\ &= 4\pi (n+2) \left( \frac{r'^{n+2}}{n+2} \right) (2)(2\pi) = \boxed{4\pi r^{n+2}} \quad (n \neq 2).\end{aligned}$$

So the divergence theorem is satisfied for  $n \neq -2$  so far. For  $n = -2$ ,

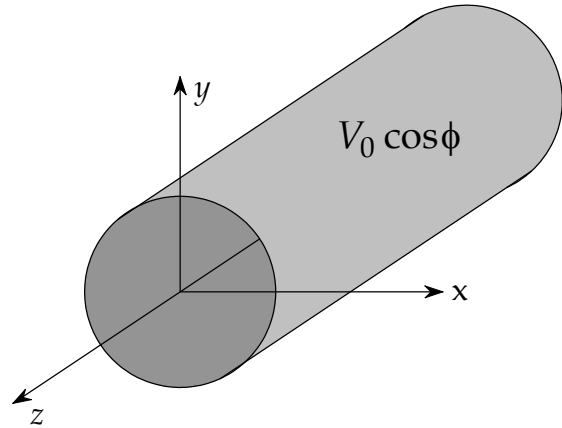
$$\oint r^{-2} \hat{r} \cdot d\mathbf{a} = \int r^{-2} r^2 \sin\theta d\theta d\phi = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = \boxed{4\pi} ,$$

$$\int \nabla \cdot (r'^n \hat{r}') d\tau' = \int 4\pi \delta^3(\mathbf{r}) r'^2 dr' d\theta' d\phi' = \boxed{4\pi} ,$$

because the integration volume contains the origin. Thus the divergence theorem is satisfied here, too, and our solution for part a is correct.

### Problem 2 (40 points)

An infinite cylindrical tube, with radius  $R$ , thin walls, and axis lying along the  $z$  direction, is held at a potential of  $V = V_0 \cos\phi$ . A small section of this tube is shown at right.



- a. Write down Laplace's equation in the coordinate system appropriate to this geometry, and indicate which term drops out because of the independence of the potential on the corresponding coordinate. Then separate the equation into two ordinary differential equations.

This problem is much like Griffiths problems 3.23 and 3.24, done on homework #6. In fact the first two parts are the same as 3.23.

The cylinder is infinite, and the boundary conditions are independent of  $z$ , so

$$\begin{aligned} \nabla^2 V &= \frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\partial V}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \\ &= \frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\partial V}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 V}{\partial \phi^2} \\ &= 0 . \end{aligned}$$

Let  $V(s, \phi) = S(s)\Phi(\phi)$ ; then

$$\frac{\Phi}{s} \frac{d}{ds} \left( s \frac{dS}{ds} \right) + \frac{S}{s^2} \frac{d^2 \Phi}{d\phi^2} = 0 \quad \text{Multiply through by } s^2/S\Phi :$$

$$\frac{s}{S} \frac{d}{ds} \left( s \frac{dS}{ds} \right) + \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = 0 = m^2 - m^2$$

$$\Rightarrow \boxed{\frac{d^2 \Phi}{d\phi^2} = -m^2 \Phi \quad , \quad s \frac{d}{ds} \left( s \frac{dS}{ds} \right) = m^2 S} .$$

- b. Solve the resulting equations for the potential, giving as your answer the most general solution. Hint: treat the case in which the separation constants are zero, separately.

$$\frac{d^2 \Phi}{d\phi^2} = -m^2 \Phi \Rightarrow \Phi = A \cos m\phi + B \sin m\phi \quad ,$$

except at  $m = 0$ , for which direct integration gives

$$\frac{d^2 \Phi}{d\phi^2} = 0 \Rightarrow \Phi = A'm + B' \quad .$$

But  $A'$  must be zero, because the solution has to be periodic ( $\Phi(\phi + 2\pi) = \Phi(\phi)$ ).

$$s \frac{d}{ds} \left( s \frac{dS}{ds} \right) = m^2 S \Rightarrow S = Cs^m + Ds^{-m} \quad ,$$

except at  $m = 0$ , which we can again integrate directly:

$$s \frac{d}{ds} \left( s \frac{dS}{ds} \right) = 0 \Rightarrow S = C' \ln s + D' \quad .$$

The general solution is a linear combination of all of the particular solutions:

$$V(s, \phi) = C_0 \ln s + D_0 + \sum_{m=1}^{\infty} (C_m s^m + D_m s^{-m}) (A_m \cos m\phi + B_m \sin m\phi) \quad .$$

If you didn't remember the particular solutions, you could derive them quickly; in the first case by using the substitution  $u = d\Phi/d\phi$  and integrating twice, and in the second by trial of  $S = Ks^n$ . See the solutions for Homework #6 for details.

- c. From your answer to part b, obtain a solution for the potential everywhere inside the cylindrical tube, by applying the boundary conditions and solving explicitly for any unknown constants.

At  $s = 0$ ,  $V$  can't be infinite, but the term with  $r^{-m}$  will be, unless all the  $D_m = 0$ .  
 At  $s = R$ ,

$$V_0 \cos \phi = C_0 \ln s + D_0 + \sum_{m=1}^{\infty} C_m R^m (A_m \cos m\phi + B_m \sin m\phi) .$$

Clearly all the constants have to be zero except for  $C_1$  and  $A_1$ , those being the ones with the term that exactly matches the  $\cos \phi$  angular dependence of the boundary condition:

$$V_0 \cos \phi = C_1 R A_1 \cos \phi \Rightarrow A_1 C_1 = \frac{V_0}{R}, \text{ so}$$

$$V(s, \phi) = V_0 \frac{s}{R} \cos \phi .$$

If you didn't notice that at a glance, you would have found it out by using Fourier's Trick, as follows. First multiply through by  $\cos n\phi$  and integrate over  $\phi$ :

$$\begin{aligned} \int_0^{2\pi} V_0 \cos \phi \cos n\phi d\phi &= \int_0^{2\pi} (C_0 \ln s + D_0) \cos n\phi d\phi \\ &+ \sum_{m=1}^{\infty} C_m R^m \left( A_m \int_0^{2\pi} \cos m\phi \cos n\phi d\phi + B_m \int_0^{2\pi} \sin m\phi \cos n\phi d\phi \right) \\ V_0 \pi \delta_{1n} &= 0 + \sum_{m=1}^{\infty} C_m R^m (A_m \pi \delta_{mn} + 0) = \pi C_n R^n A_n \\ &\Rightarrow A_1 C_1 = \frac{V_0}{R}, \text{ all other } A_n C_n = 0. \end{aligned}$$

Then multiply through by  $\sin n\phi$  and integrate over  $\phi$ :

$$\begin{aligned} \int_0^{2\pi} V_0 \cos \phi \sin n\phi d\phi &= \int_0^{2\pi} (C_0 \ln s + D_0) \sin n\phi d\phi \\ &+ \sum_{m=1}^{\infty} C_m R^m \left( A_m \int_0^{2\pi} \cos m\phi \sin n\phi d\phi + B_m \int_0^{2\pi} \sin m\phi \sin n\phi d\phi \right) \\ 0 &= 0 + \sum_{m=1}^{\infty} C_m R^m (0 + B_m \pi \delta_{mn}) = \pi C_n R^n B_n \\ &\Rightarrow B_n C_n = 0. \end{aligned}$$

This leaves

$$V(s, \phi) = C_0 \ln s + D_0 + V_0 \frac{s}{R} \cos \phi \quad .$$

Finally, at  $s = R$ ,  $\phi = \pi/2$ , we have

$$0 = C_0 \ln s + D_0 + 0 \Rightarrow C_0 = D_0 = 0 \quad ,$$

whence, again,

$$V(s, \phi) = V_0 \frac{s}{R} \cos \phi \quad .$$

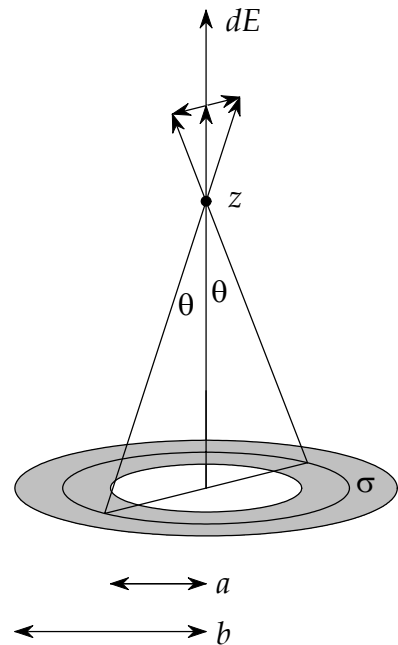
You'll get full credit either way.

### Problem 3 (40 points)

- a. Calculate the electric field a distance  $z$  above the center of a charged washer that has charge per unit area  $\sigma$ , outer radius  $b$ , and inner radius  $a$ . Hint: don't calculate the potential first.

In cylindrical coordinates, consider charge elements at radius  $s$  on opposite sides of the axis from one another. The  $s$  components of their contribution to the electric field cancel, and the  $z$  components add:

$$\begin{aligned} dE &= \hat{z} 2 \frac{\sigma s ds d\phi}{r^2} \cos \theta \\ &= \hat{z} 2 \frac{\sigma s ds d\phi}{s^2 + z^2} \frac{z}{\sqrt{s^2 + z^2}} \end{aligned}$$



$$\begin{aligned}
 E &= \sigma z \hat{z} \int_0^\pi d\phi \int_a^b \frac{2s ds}{(s^2 + z^2)^{3/2}} \\
 &= \pi \sigma z \hat{z} \int_{a^2+z^2}^{b^2+z^2} u^{-3/2} du = -2\pi \sigma z \hat{z} u^{-1/2} \Big|_{a^2+z^2}^{b^2+z^2} \\
 &= \boxed{2\pi \sigma z \hat{z} \left( \frac{1}{\sqrt{a^2 + z^2}} - \frac{1}{\sqrt{b^2 + z^2}} \right)}.
 \end{aligned}$$

- b. Calculate the electric field a distance  $z$  above the center of a charged tube that has charge density  $\rho$ , outer radius  $b$ , inner radius  $a$ , and length  $b$ . Hint: use the result from part a.

Carve the tube up into infinitesimal washers with thickness  $dz$ , and use the previous result, substituting  $\rho dz$  for  $\sigma$ :

$$dE = 2\pi \rho z' dz' \hat{z} \left( \frac{1}{\sqrt{a^2 + z'^2}} - \frac{1}{\sqrt{b^2 + z'^2}} \right).$$

Then integrate over the thickness of the tube. We can do this from the bottom to the top ( $z + b$  to  $z$ ):

$$\begin{aligned}
 E &= \pi \rho \hat{z} \int_{z+b}^z \left( \frac{2z' dz'}{\sqrt{a^2 + z'^2}} - \frac{2z' dz'}{\sqrt{b^2 + z'^2}} \right) = \pi \rho \hat{z} \int_{a^2+(z+b)^2}^{a^2+z^2} u^{-1/2} du - \pi \rho \hat{z} \int_{b^2+(z+b)^2}^{b^2+z^2} v^{-1/2} dv \\
 &= 2\pi \rho \hat{z} \left( u^{1/2} \Big|_{a^2+(z+b)^2}^{a^2+z^2} - v^{1/2} \Big|_{b^2+(z+b)^2}^{b^2+z^2} \right) \\
 &= \boxed{2\pi \rho \hat{z} \left( \sqrt{a^2 + z^2} + \sqrt{b^2 + (z+b)^2} - \sqrt{b^2 + z^2} - \sqrt{a^2 + (z+b)^2} \right)}.
 \end{aligned}$$

