Today in Physics 217: vector analysis

Vectors:
- have direction
- have magnitude

Vector operations include:
- vector addition
- vector multiplication by a scalar
- the dot product
- the cross product

Vector components
Vector transformation
Second-rank tensors

Vector operations

Vector addition:
- Adding two vectors produces a third vector:
  \[ A + B = C \]
- Vector addition is commutative:
  \[ A + C = B + A \]
- Vector subtraction is equivalent to adding the opposite of a vector:
  \[ A - B = A + (-B) \]

Vector multiplication by a scalar:
- The result of vector multiplication by a scalar is a vector.
- The magnitude of the resulting vector is the product of the magnitude of the scalar and the magnitude of the vector.
- The direction of the resulting vector is the same as the direction of the original vector if \( a > 0 \) and opposite to the direction of the original vector if \( a < 0 \).

Vector operations (continued)

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Vector multiplication is distributive:
- \( a \cdot (A + B) = aA + aB \)
Vector operations (continued)

The dot product (scalar product):
- The results of the dot product is a scalar:
  \[ \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos \theta = \mathbf{A} \mathbf{B} \cos \theta \]
- The dot product is commutative:
  \[ \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \]
- The dot product is distributive:
  \[ \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \]

Cross product (vector product):
- The result of the cross product is a vector perpendicular to the two original vectors.
  - Magnitude:
    \[ |\mathbf{A} \times \mathbf{B}| = |\mathbf{A}| |\mathbf{B}| \sin \theta \]
  - Direction: use right-hand rule
- The cross product is not commutative:
  \[ \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \]
- The cross product is distributive:
  \[ \mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \]

Vector components

A vector can be identified by specifying its three Cartesian components:

\[ \mathbf{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \]

Unit vectors

Vector operations:
- To add vectors, add like components.
- To multiply a vector by a scalar, multiply each component.
Vector components (continued)

To calculate the dot product of two vectors, multiply like components and add:
\[ \mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \]

To calculate the cross product of two vectors, evaluate the following determinant:
\[
\mathbf{A} \times \mathbf{B} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
A_x & A_y & A_z \\
B_x & B_y & B_z
\end{vmatrix}
\]

Vector transformation

The components of a vector depend on the choice of the coordinate system.
Different coordinate systems will produce different components for the same vector.
The choice of coordinate system being used can significantly change the complexity of problems in electrodynamics.

Vector transformation (continued)

The coordinates of vector \( \mathbf{A} \) in coordinate system \( S \) are related to the coordinates of vector \( \mathbf{A} \) in coordinate system \( S' \):
\[
\begin{pmatrix}
A' \phi \\
A' \theta \\
A' \phi
\end{pmatrix} = \begin{pmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
A \phi \\
A \theta \\
A \phi
\end{pmatrix}
\]
The rotation considered here leaves the \( x \) axis untouched. The \( x \) coordinate of vector \( \mathbf{A} \) will thus not change:
\[
\begin{pmatrix}
A' \phi \\
A' \theta \\
A' \phi
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{pmatrix} \begin{pmatrix}
A \phi \\
A \theta \\
A \phi
\end{pmatrix} = \mathbf{R} \mathbf{A}
\]
Vector transformation (continued)

Coordinate transformation resulting from a rotation around an arbitrary axis can be written as:

\[
\begin{pmatrix}
A_x' \\
A_y' \\
A_z'
\end{pmatrix} =
\begin{pmatrix}
R_{xx} & R_{xy} & R_{xz} \\
R_{yx} & R_{yy} & R_{yz} \\
R_{zx} & R_{zy} & R_{zz}
\end{pmatrix}
\begin{pmatrix}
A_x \\
A_y \\
A_z
\end{pmatrix} = R_{xx}A_x + R_{xy}A_y + R_{xz}A_z
\]

or, more compactly, with \( x \) denoted as 1, \( y \) as 2, \( z \) as 3:

\[A'_j = \sum_{j=1}^{3} R_{ij}A_j\]

Vector transformation (continued)

The rotation matrix \( R \) is an example of a unitary transformation: one that does not change the magnitude of the object on which it operates:

\[A' = \hat{R} \cdot A \quad \text{and} \quad A'' = A.\]

If \( R \) is unitary, then \[\sum_{j=1}^{3} R_{ij}R_{jk} = \delta_{ik},\]

where \( \delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases} \) (the Kronecker delta),

as you will make plausible in this week’s homework.

Second-rank tensors

Vectors are first-rank tensors, having three independent components that can be represented by a column matrix. An object \( T \) with nine independent components that can multiply a vector and produce a vector result,

\[B = \vec{T} \cdot A\]

are called second-rank tensors. They behave as follows under rotations:

\[T'_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} R_{ik}R_{jl}T_{kl}\]