Today in Physics 217: vector analysis

Vectors:
- ❑ have direction
- ❑ have magnitude

Vector operations include
- ❑ vector addition
- ❑ vector multiplication by a scalar
- ❑ the dot product
- ❑ the cross product

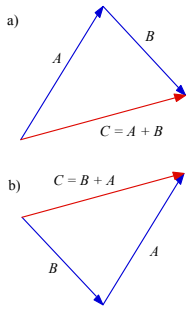
Vector components

Vector transformation

Second-rank tensors

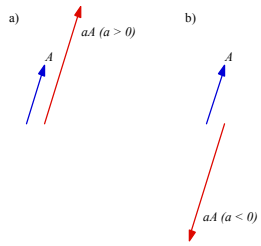
Vector operations

Vector addition:
- ❑ Adding two vectors produces a third vector:
  $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{C}$
- ❑ Vector addition is commutative:
  $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{C}=\boldsymbol{B}+\boldsymbol{A}$
- ❑ Vector subtraction is equivalent to adding the opposite of a vector:
  $\boldsymbol{A}-\boldsymbol{B}=\boldsymbol{A}+(-\boldsymbol{B})$



Vector operations (continued)

Vector multiplication by a scalar:
- ❑ The result of vector multiplication by a scalar is a vector.
- ❑ The magnitude of the resulting vector is the product of the magnitude of the scalar and the magnitude of the vector.
- ❑ The direction of the resulting vector is the same as the direction of the original vector if $a > 0$ and opposite to the direction of the original vector if $a < 0$.
- ❑ Vector multiplication is distributive:
  $a(\boldsymbol{A}+\boldsymbol{B})=a\boldsymbol{A}+a\boldsymbol{B}$



Vector operations (continued)

The dot product (scalar product):

- The results of the dot product is a scalar:

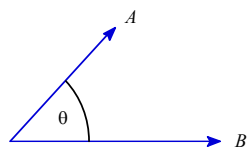
$$\boldsymbol{A}\cdot\boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\cos\theta = AB\cos\theta$$

- The dot product is commutative:

$$\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{A}$$

- The dot product is distributive:

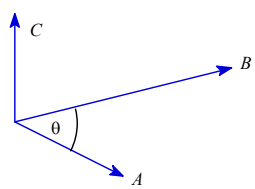
$$\boldsymbol{A}\cdot(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{A}\cdot\boldsymbol{B}+\boldsymbol{A}\cdot\boldsymbol{C}$$



Vector operations (continued)

Cross product (vector product):

- The result of the cross product is a vector perpendicular to the two original vectors.
  - Magnitude: $|\boldsymbol{C}| = |\boldsymbol{A}\times\boldsymbol{B}| = AB\sin\theta$
  - Direction: use right-hand rule
- The cross product is not commutative: $\boldsymbol{A}\times\boldsymbol{B} = -\boldsymbol{B}\times\boldsymbol{A}$
- The cross product is distributive: $\boldsymbol{A}\times(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{A}\times\boldsymbol{B}+\boldsymbol{A}\times\boldsymbol{C}$



Vector components

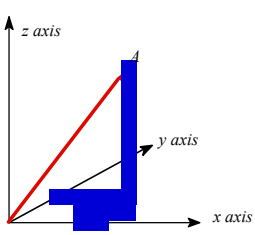
A vector can be identified by specifying its three Cartesian components:

$$\boldsymbol{A} = A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}}$$

Unit vectors

Vector operations:

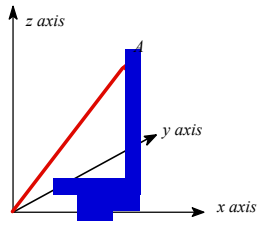
- To add vectors, add like components.
- To multiply a vector by a scalar, multiply each component.



Vector components (continued)

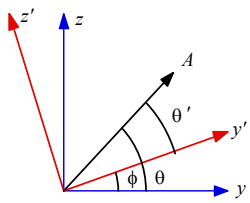
- To calculate the dot product of two vectors, multiply like components and add:
  $\boldsymbol{A}\cdot\boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z$
- To calculate the cross product of two vectors, evaluate the following determinant:

$$\boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



Vector transformation

- The components of a vector depend on the choice of the coordinate system.
- Different coordinate system will produce different components for the same vector.
- The choice of coordinate system being used can significantly change the complexity of problems in electrodynamics.



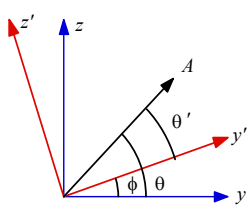
Vector transformation (continued)

The coordinates of vector $\boldsymbol{A}$ in coordinate system S are related to the coordinates of vector $\boldsymbol{A}$ in coordinate system S′:

$$\begin{pmatrix} A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}$$

The rotation considered here leaves the x axis untouched. The x coordinate of vector A will thus not change:

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \equiv \boldsymbol{R}\cdot\boldsymbol{A}$$



Vector transformation (continued)

Coordinate transformation resulting from a rotation around an arbitrary axis can be written as:

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} R_{xx}A_x + R_{xy}A_y + R_{xz}A_z \\ R_{yx}A_x + R_{yy}A_y + R_{yz}A_z \\ R_{zx}A_x + R_{zy}A_y + R_{zz}A_z \end{pmatrix}$$

or, more compactly, with x denoted as 1, y as 2, z as 3:

$$A_i' = \sum_{j=1}^{3} R_{ij} A_j$$

Vector transformation (continued)

The rotation matrix R is an example of a **unitary** transformation: one that does not change the magnitude of the object on which it operates:

$$\boldsymbol{A}' = \ddot{\boldsymbol{R}} \cdot \boldsymbol{A} \quad \text{and} \quad A' = A.$$

If R is unitary,then $\sum_{i=1}^{3} R_{ij} R_{ik} = \delta_{jk}$,

where $\delta_{jk} = \begin{cases} 1 \text{ if } j = k \\ 0 \text{ otherwise} \end{cases}$ (the Kroneker delta),

as you will make plausible in this week's homework.

Second-rank tensors

Vectors are first-rank tensors, having three independent components that can be represented by a column matrix. An object $\ddot{\boldsymbol{T}}$ with nine independent components that can multiply a vector and produce a vector result,

$$\boldsymbol{B} = \ddot{\boldsymbol{T}} \cdot \boldsymbol{A}$$

are called second-rank tensors. They behave as follows under rotations:

$$T_{ij}' = \sum_{k=1}^{3} \sum_{l=1}^{3} R_{ik} R_{jl} T_{kl}$$