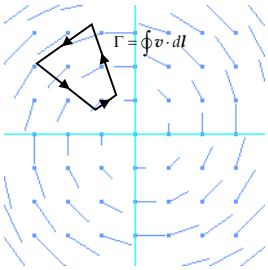
Today in Physics 217: vector integrals

Three useful generalizations of the fundamental theorem of calculus:

- Gradient theorem
- Gauss' divergence theorem
- Stokes' curl theorem



Integral vector calculus

Fundamental theorem of calculus for a function of one variable:

$$\int_a^b \frac{df(x)}{dx} dx = f(b) - f(a)$$

In vector calculus, there are three different kinds of derivatives – gradient, divergence and curl – so there are three different analogues of the fundamental theorem of calculus:

- the **gradient theorem**:

$$\int_C \nabla T \cdot d\boldsymbol{l} = T(\boldsymbol{b}) - T(\boldsymbol{a})$$

where the integral is taken along the curve *C*, and ***a*** and ***b*** are the position vectors of the endpoints of *C*.

Integral vector calculus (continued)

- **Stokes' theorem**, for curls:

$$\int_S (\nabla \times \boldsymbol{v}) d\boldsymbol{a} = \oint_C \boldsymbol{v} \cdot d\boldsymbol{l}$$

where the integral on the left is carried out over a surface *S*, and that on the right is carried out all the way around the curve *C* that bounds *S*.

- And (Gauss') **divergence theorem**:

$$\int_V (\nabla \cdot \boldsymbol{v}) d\tau = \oint_S \boldsymbol{v} \cdot d\boldsymbol{a}$$

where the integral on the left is carried out over a volume *V*, and that on the right over the surface *S* that bounds *V*.

Illustrating these theorems one by one…

Gradient theorem

$$\int_C \nabla T \cdot d\boldsymbol{l} = T(\boldsymbol{b}) - T(\boldsymbol{a})$$

- The left-hand side is a line integral. It is evaluated by choosing a specific path from $\boldsymbol{a}$ to $\boldsymbol{b}$.
- The theorem ensures that the result is independent of the path chosen. (So choose one that makes the integral easy…) This is not true of arbitrary vector functions: only gradients have this property.
- The line integral of the gradient of T around a closed loop is zero:

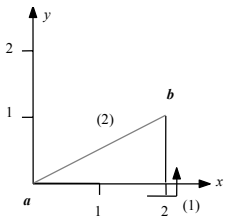
$$\oint \nabla T \cdot d\boldsymbol{l} = T(\boldsymbol{a}) - T(\boldsymbol{a}) = 0$$

Example illustration of the gradient theorem

Example 1.6 in Griffiths: Verify the gradient theorem for this function and these bounds,

$$T = xy^2$$
$$a = (0 \quad 0 \quad 0)$$
$$b = (2 \quad 1 \quad 0)$$

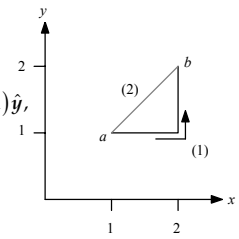
using both of the paths shown at right.



Not everything is a gradient.

Problem 1.30 in Griffiths: calculate $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{v} \cdot d\boldsymbol{l}$, from $\boldsymbol{a} = (1 \quad 1 \quad 0)$ to $\boldsymbol{b} = (2 \quad 2 \quad 0)$, for $v = y^2\hat{\boldsymbol{x}} + 2x(y+1)\hat{\boldsymbol{y}}$, using both paths shown at right. Is the result independent of path?

Answer: (1) 11, (2) 10, so the result depends on path; this function isn't a gradient.



Divergence theorem

$$\int_V (\nabla \cdot \boldsymbol{v})\, d\tau = \oint_S \boldsymbol{v} \cdot d\boldsymbol{a}$$

- The right-hand side is the **flux** of the vector function $\boldsymbol{v}$ through the surface S. For example, if $\boldsymbol{v}$ were the velocity of a fluid, this would tell us the rate of flow of the fluid through the surface.
- The left hand side is the sum of all the **sources** of $\boldsymbol{v}$ within the volume. Again, in the example of fluid flow, this would be the sum of the output of the "faucets," which in turn are places where the divergence of $\boldsymbol{v}$ is high.

Proof – based on the nice one in Ch. 2 of Purcell – is provided on the chalkboard.

Stokes' theorem

$$\int_S (\nabla \times \boldsymbol{v})\, d\boldsymbol{a} = \oint_C \boldsymbol{v} \cdot d\boldsymbol{l}$$

- The right-hand side is the **circulation** of the vector function $\boldsymbol{v}$ on curve C. Again the name comes from the fluid-flow analogy.
- The left-hand side is the sum of the **sources** of circulation with the area bounded by C. If high divergence corresponds to a productive water faucet within V, high curl corresponds to a good stirring-rod within S.

Again there's a nice proof emphasizing this analogy, which I'll show on the blackboard.

- Note: the orientation of the surface vector and the direction of the integration along the boundary should be consistent with the right-hand rule.