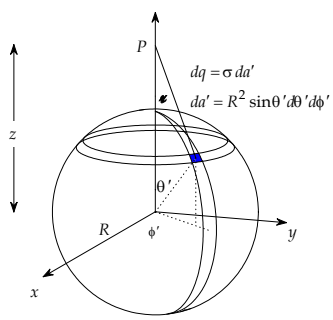
Today in Physics 217: work and energy in electrostatics

- Last spherical example: potential from a uniformly-charged spherical shell
- Work and potential energy
- Electrostatic potential energy
- Inconsistency?
- Non-superposition of potential energy

Potential from a uniformly-charged spherical shell

Griffiths, example 2.7:
What is the electric potential a distance z away from the center of a spherical shell with radius R and uniform surface charge density σ?

Please accept my apologies for doing an example that's worked out in the book. But it goes with the last few examples, and I'd like to present them as a set. Compare especially to proble 2.7, done in class Friday.



Potential from a uniformly-charged spherical shell (continued)

As we saw on Friday,

$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta'$$

so

$$V(z) = \int \frac{\sigma\, da'}{\mathscr{r}} = \sigma \int_0^{2\pi} d\phi' \int_0^{\pi} \frac{R^2 \sin\theta'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}\, d\theta'$$

The first integral is trivial: it just comes out to 2π. For the second, substitute

$$w = \cos\theta', \quad dw = \sin\theta' d\theta', \quad w = 1 \rightarrow -1:$$

so

$$V(z) = 2\pi\sigma R^2 \int_{-1}^{1} \frac{dw}{\sqrt{R^2 + z^2 - 2Rzw}}$$

Potential from a uniformly-charged spherical shell (continued)

Then, substitute

$$u = R^2 + z^2 - 2Rzw, \quad du = -2Rzdw,$$

$$u = R^2 + z^2 + 2Rz \to R^2 + z^2 - 2Rz$$

to get

$$V(z) = \frac{\pi\sigma R}{z}\int_{R^2+z^2-2Rz}^{R^2+z^2+2Rz} u^{-1/2}du = \frac{\pi\sigma R}{z}\left[2\sqrt{u}\right]_{R^2+z^2-2Rz}^{R^2+z^2+2Rz}$$

$$= \frac{2\pi\sigma R}{z}\left(\sqrt{R^2+z^2+2Rz} - \sqrt{R^2+z^2-2Rz}\right)$$

Almost at the answer already! But, as before, we must take positive square roots here, so note that

$$R^2 + z^2 \pm 2Rz = (z \pm R)^2$$

Potential from a uniformly-charged spherical shell (continued)

so that we get $V(z) = \frac{2\pi\sigma R}{z}\left(|z+R| - |z-R|\right)$

Two cases: z larger than, or smaller than, R. (P outside, inside)

- **Larger (outside)**:

$$V(z) = \frac{2\pi\sigma R}{z}(z + R - z + R) = \frac{4\pi R^2\sigma}{z} = \frac{Q}{z}$$

Behaves like a point charge at the sphere's center.

- **Smaller (inside)**: means $|z-R| = R - z$, so

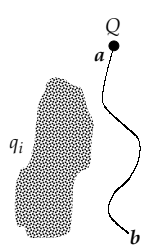
$$V(z) = \frac{2\pi\sigma R}{z}(z + R - R + z) = 4\pi R\sigma = \frac{Q}{R}$$

Constant potential (thus zero field) inside spherical shell.

Much easier than Friday's calculation!

Work done in motion of a test charge

Move a test charge Q around in the field of a collection of other charges. How much work is done moving it from $\boldsymbol{a}$ to $\boldsymbol{b}$?



The force exerted on the charge is $\boldsymbol{F} = Q\boldsymbol{E}$; the force *we* need to exert, by Newton's third law, is - $Q\boldsymbol{E}$. So the work *we* do is

$$W = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{F}\cdot d\boldsymbol{l} = -Q\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E}\cdot d\boldsymbol{l} = Q\left[V(\boldsymbol{b}) - V(\boldsymbol{a})\right]$$

(independent of path). Corollary: the work required to bring charge Q to point $\mathcal{P}$ from infinity is

$$W = Q\left[V(\mathcal{P}) - V(\infty)\right] = QV(\mathcal{P})$$

Electrostatic potential energy

To obtain the **potential energy** of an assembly of charges, bring them from infinity in one by one, and calculate the work done. Consider assembling the charge distribution above: a bunch of point charges, q_i.

Bring in the first one: $W_1 = 0$

Bring in the second one: $W_2 = q_2\left(\frac{q_1}{\mathscr{r}_{12}}\right)$

And the third one: $W_3 = q_3\left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}}\right)$

And the fourth: $W_4 = q_4\left(\frac{q_1}{\mathscr{r}_{14}} + \frac{q_2}{\mathscr{r}_{24}} + \frac{q_3}{\mathscr{r}_{34}}\right)$

Electrostatic potential energy (continued)

So far, for the first four charges, the total work is

$$W = q_2\left(\frac{q_1}{\mathscr{r}_{12}}\right) + q_3\left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}}\right) + q_4\left(\frac{q_1}{\mathscr{r}_{14}} + \frac{q_2}{\mathscr{r}_{24}} + \frac{q_3}{\mathscr{r}_{34}}\right)$$

$$= \frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_2 q_3}{\mathscr{r}_{23}} + \frac{q_1 q_4}{\mathscr{r}_{14}} + \frac{q_2 q_4}{\mathscr{r}_{24}} + \frac{q_3 q_4}{\mathscr{r}_{34}}$$

Evidently, for N charges, we'd get

$$W = \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{2}\sum_{i=1}^{N} q_i \sum_{\substack{j=1,\\ i\neq j}}^{N} \frac{q_j}{\mathscr{r}_{ij}} = \frac{1}{2}\sum_{i=1}^{N} q_i V(\mathcal{P}_i)$$

(Count each pair of charges just once)

Electrostatic potential energy (continued)

If the collection of charges is finite and continuous $(N \to \infty)$, this becomes

$$\boxed{W = \frac{1}{2}\int_{\mathcal{V}} dq\, V = \frac{1}{2}\int_{\mathcal{V}} \rho V d\tau}$$

As usual, we'd also have

$$W = \frac{1}{2}\int_{\mathcal{S}} \sigma V da \qquad W = \frac{1}{2}\int_{\mathcal{C}} \lambda V d\ell$$

for surface- and line-charge distributions.

Eliminate density and potential from these expressions, in favor of the electric field, with

$$\rho = \frac{1}{4\pi}\nabla\cdot\boldsymbol{E} \qquad \nabla V = -\boldsymbol{E}$$

Electrostatic potential energy (continued)

Recall also that $\nabla\cdot(f\boldsymbol{A}) = (\nabla\cdot\boldsymbol{A})f + \boldsymbol{A}\cdot(\nabla f)$:

$$W = \frac{1}{8\pi}\int_{\mathcal{V}}(\nabla\cdot\boldsymbol{E})V d\tau = \frac{1}{8\pi}\int_{\mathcal{V}}\left[\nabla\cdot(V\boldsymbol{E}) - \boldsymbol{E}\cdot\nabla V\right]d\tau$$

$$= \frac{1}{8\pi}\int_{\mathcal{V}}\left[\nabla\cdot(V\boldsymbol{E}) + E^2\right]d\tau = \frac{1}{8\pi}\left[\oint_{\mathcal{S}} V\boldsymbol{E}\cdot d\boldsymbol{a} + \int_{\mathcal{V}} E^2 d\tau\right] .$$

Divergence theorem

Electrostatic potential energy (continued)

If we extend the integration region $\mathcal{V}$ to include all of space, and the charge distribution is finite in extent, then $\boldsymbol{E}$ and V approach zero at the surface $\mathcal{S}$ (which "surrounds infinity").
Note that

$$\lim_{r\to\infty} E \propto \frac{1}{r^2} \qquad \lim_{r\to\infty} V \propto \frac{1}{r} \qquad \lim_{r\to\infty} A \propto r^2$$

$$\therefore \lim_{r\to\infty}\oint_{\mathcal{S}} V\boldsymbol{E}\cdot d\boldsymbol{a} \propto \frac{1}{r} = 0$$

Thus

$$\boxed{W = \frac{1}{8\pi}\int_{\substack{\text{all}\\\text{space}}} E^2 d\tau} = \frac{\varepsilon_0}{2}\int_{\substack{\text{all}\\\text{space}}} E^2 d\tau \quad \text{in MKS.}$$

An inconsistency?

Consider a point charge, for which the charge density is

$$\rho = q\delta^3(\boldsymbol{r})$$

How much work is involved in assembly of this charge distribution? On the one hand,

$$W = \frac{1}{2}\int \rho V d\tau = \frac{q}{2}\int \delta^3(\boldsymbol{r})\frac{q}{r}r^2\sin\theta\, dr d\theta\, d\phi = 2\pi q^2\int_0^\infty \delta(r) r dr = 0$$

but on the other,

$$W = \frac{1}{8\pi}\int E^2 d\tau = \frac{q^2}{2}\int_0^\infty \frac{1}{r^4}r^2 dr = -\frac{q^2}{6}\frac{1}{r^3}\bigg|_0^\infty \to \infty$$

An inconsistency? (continued)

Reason: related to the troublesome divergence of $\hat{\boldsymbol{r}}/r^2$ (cf. problems 1.16, 1.38). Restore the surface integral to the expression of potential energy in terms of field:

$$W = \frac{1}{8\pi}\left[\oint_{\mathcal{S}} V\boldsymbol{E}\cdot d\boldsymbol{a} + \int_{\mathcal{V}} E^2 d\tau\right] = \frac{1}{8\pi}\int_{\mathcal{V}}\left[\nabla\cdot(V\boldsymbol{E}) + E^2\right]d\tau \quad .$$

For our point charge,

$$\nabla\cdot(V\boldsymbol{E}) + E^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{q}{r}\frac{q}{r^2}\right) + \frac{q^2}{r^4} = \frac{q^2}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \frac{q^2}{r^4}$$

$$= -\frac{q^2}{r^4} + \frac{q^2}{r^4} = 0 \quad \Rightarrow \quad W = \int\left[\nabla\cdot(V\boldsymbol{E}) + E^2\right]d\tau = 0$$

Not, therefore, inconsistent. But use $W = 1/2\int E^2 d\tau$ with care, and keep that surface integral in mind.

Non-superposition of potential energy

As you know, forces, electric fields, and electric potentials obey the principle of superposition. Potential energy does not. Consider:

$$W = \frac{1}{8\pi}\int E^2 d\tau = \frac{1}{8\pi}\int(\boldsymbol{E}_1 + \boldsymbol{E}_2 + \ldots)\cdot(\boldsymbol{E}_1 + \boldsymbol{E}_2 + \ldots)d\tau$$

$$\neq \frac{1}{8\pi}\int\left(E_1^2 + E_2^2 + \ldots\right)d\tau = W_1 + W_2 + \ldots \quad ,$$

because cross terms such as

$$\frac{1}{8\pi}\int 2\boldsymbol{E}_1\cdot\boldsymbol{E}_2 d\tau \quad , \quad \frac{1}{8\pi}\int 2\boldsymbol{E}_1\cdot\boldsymbol{E}_3 d\tau \quad ,\ldots$$

are not necessarily zero.