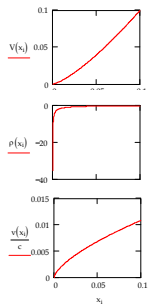
Today in Physics 217: boundary conditions and electrostatic boundary-value problems

- Boundary conditions in electrostatics
- Simple solution of Poisson's equation as a boundary-value problem: the space-charge limited vacuum diode



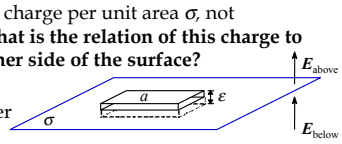
Electrostatic boundary-value problems

We have encountered several differential equations relating the field, scalar potential, and charge density, that we will in general want to solve for $\boldsymbol{E}$ or V. But the solutions aren't unique unless enough constraints – **boundary values** – are supplied.

- If the potential, or components of the field, are specified on some surface or line (e.g. "equipotentials), a unique solution of the differential equation(s) for V or $\boldsymbol{E}$ in the rest of the volume or area will be obtained. This will usually be how solutions of Laplace's equation will proceed.
- Specification of the charge density results in specification of boundary conditions on field and potential, as follows. (This often comes in when one solves the Poisson equation.)

Surface charges and electric-field boundary conditions

Consider a surface with charge per unit area σ, not necessarily constant. **What is the relation of this charge to the electric field on either side of the surface?**



Zoom in on an area small enough to consider flat, and draw a Gaussian surface with planar symmetry that has sides ε $(\to 0)$ perpendicular to the surface. Then, since the flux through the sides is negligible,

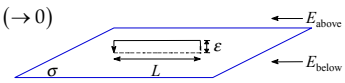
$$\oint \boldsymbol{E}\cdot d\boldsymbol{a} = 4\pi Q_{\text{enclosed}}$$

$$\left(E_{\perp,\text{ above}} - E_{\perp,\text{ below}}\right)a = 4\pi\sigma\, a \quad \Rightarrow \quad \boxed{\left(E_{\perp,\text{ above}} - E_{\perp,\text{ below}}\right) = 4\pi\sigma}$$

The charge sheet makes a discontinuity of $4\pi\sigma$ in $E_\perp$.

Surface charges and electric-field boundary conditions (continued)

Now consider a loop, $\varepsilon\ (\to 0)$ by L, perpendicular to the surface, bisected by the surface, and small enough that the surface is flat over its dimensions. In a line integral of field along this loop the contribution of the short sides is negligibly small, so



$$\oint \boldsymbol{E}\cdot d\boldsymbol{l} = 0$$

$$\left(E_{\parallel,\,\text{above}} - E_{\parallel,\,\text{below}}\right)L = 0 \quad \Rightarrow \quad \boxed{E_{\parallel,\,\text{above}} - E_{\parallel,\,\text{below}} = 0}$$

The parallel component of $\boldsymbol{E}$ is continuous across the sheet of charge.

Surface charges and electric-potential boundary conditions

Implications of this on V:

- For two points $\boldsymbol{a}$ and $\boldsymbol{b}$, $\pm\varepsilon/2$ from the surface,

$$\boxed{V_{\text{above}} - V_{\text{below}} = -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E}\cdot d\boldsymbol{l} \xrightarrow[\varepsilon\to 0]{} 0}$$

  i.e. V is continuous across the charged sheet.

- Consider the gradient of V.

$$\left(\nabla V_{\text{above}} - \nabla V_{\text{below}}\right)_{\perp} = -\left(E_{\perp,\,\text{above}} - E_{\perp,\,\text{below}}\right) = -4\pi\sigma$$

  Or, with n as the coordinate axis perpendicular to the surface at this point,

$$\left(\nabla V_{\text{above}} - \nabla V_{\text{below}}\right)\cdot\hat{\boldsymbol{n}} = -4\pi\sigma \quad \Rightarrow \quad \boxed{\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -4\pi\sigma}$$

Example: the space-charge limited vacuum diode

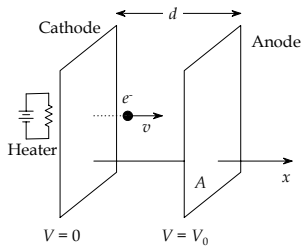
Griffiths problem ! 2.48: In a vacuum diode, electrons are boiled off a hot cathode, at potential zero, and accelerated across a gap to the anode, which is held at positive potential V_0. The cloud of moving electrons within the gap (called the space charge) quickly builds up to the point where it reduces the field at the surface of the cathode to zero. From then on, a steady current I flows between the plates.

Suppose the plates are large relative to the separation d, so that edge effects can be neglected. Then V, ρ, and v (the speed of the electrons) are functions of x alone.

(a) Solve Poisson's equation for the potential between the anode and cathode, as a function of x, V_0, and d.

(b) Show that $I = KV_0^{3/2}$, and find the constant K. This expression is called **Child's Law**.

Example: the space-charge limited vacuum diode (continued)



As soon as electrons boil off in large numbers, their presence slows down further electron ejection because they're so slow at first. (This is what is meant by "space-charge limited".) The space charge is denser near the cathode.

Although the electrons move, the space charge distribution is constant after a while, so this *is* an electrostatics problem.

Example: the space-charge limited vacuum diode (continued)

Poisson's equation, in one dimension: $\nabla^2 V = \frac{d^2V}{dx^2} = -4\pi\rho(x)$

Boundary conditions: $V(0)=0, \quad V(d)=V_0, \quad \boldsymbol{E}(0)=0$

Solution:

First, consider a charge q that makes it to where the potential is V. The field has done work $W = qV$, so

$$W = qV = \Delta(KE) = \frac{1}{2}mv^2 \quad \Rightarrow \quad v = \sqrt{2qV/m}$$

Also consider a slice of the space charge, with thickness dx:

$$dq = \rho A dx \quad \Rightarrow \quad \frac{dq}{dt} = \rho A \frac{dx}{dt} = \rho A v$$

Example: the space-charge limited vacuum diode (continued)

But $dq/dt = I$, which is constant if the charge distribution has reached a steady state, so

$$\frac{d^2V}{dx^2} = -4\pi\rho = -4\pi\frac{I}{Av} = -4\pi\frac{I}{A}\sqrt{\frac{m}{2qV}} = \beta V^{-1/2}$$

where $\beta = -4\pi\frac{I}{A}\sqrt{\frac{m}{2q}}$.

To solve, substitute $dV/dx = V'$, and multiply through by V':

$$V'\frac{d^2V}{dx^2} = \beta V^{-1/2}V'$$

$$V'\frac{dV'}{dx} = \beta V^{-1/2}\frac{dV}{dx}$$

Example: the space-charge limited vacuum diode (continued)

Integrate with respect to x:

$$\int V' \frac{dV'}{dx} dx = \beta \int V^{-1/2} \frac{dV}{dx} dx$$

$$\frac{1}{2} V'^2 = 2\beta V^{1/2} + C$$

But $V(0) = 0$, and $E(0) = 0 = -\frac{dV}{dx}(0) = -V'(0)$, so $C = 0$, and

$$V' = \frac{dV}{dx} = 2\sqrt{\beta} V^{1/4}$$

$$\int V^{-1/4} dV = 2\sqrt{\beta} \int dx$$

$$\frac{4}{3} V^{3/4} = 2\sqrt{\beta} x + D$$

Example: the space-charge limited vacuum diode (continued)

$V(0) = 0$ once again mandates that $D = 0$. Also $V(d) = V_0$, so

$$\frac{4}{3} V_0^{3/4} = 2\sqrt{\beta} d \quad \Rightarrow \quad \beta = \left(\frac{2}{3d}\right)^2 V_0^{3/2}$$

$$\Rightarrow \quad V(x) = \left(\frac{3}{2}\sqrt{\beta} x\right)^{4/3} = \left(\frac{3}{2}\left(\frac{2}{3d}\right) V_0^{3/4} x\right)^{4/3} \boxed{= V_0 \left(\frac{x}{d}\right)^{4/3}}$$

That's the solution of part (a). To work out the current (part b), start by getting the charge density and speed:

$$\rho = -\frac{1}{4\pi} \frac{d^2 V}{dx^2} = -\frac{1}{4\pi} \frac{V_0}{d^{4/3}} \frac{4}{3} \frac{1}{3} x^{-2/3} = -\frac{V_0}{9\pi \left(d^2 x\right)^{2/3}}$$

$$v = \sqrt{\frac{2qV}{m}} = \sqrt{\frac{2qV_0}{m}} \left(\frac{x}{d}\right)^{2/3}$$

Example: the space-charge limited vacuum diode (continued)

Thus

$$I = \rho A v = -\frac{V_0 A}{9\pi \left(d^2 x\right)^{2/3}} \sqrt{\frac{2qV_0}{m}} \left(\frac{x}{d}\right)^{2/3}$$

$$= -\frac{A}{9\pi d^2} \sqrt{\frac{2q}{m}} V_0^{3/2} = K V_0^{3/2},$$

where $\boxed{K = -\frac{A}{9\pi d^2} \sqrt{\frac{2q}{m}}.}$ **Child's Law**

Note that the current and voltage are not directly proportional, as in Ohm's law.