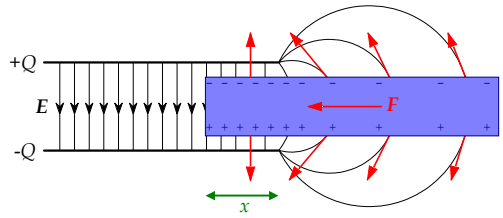
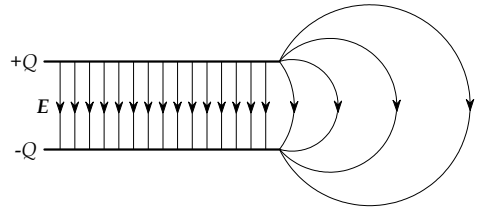
Today in Physics 217: charges in motion

- ❑ Forces on dielectrics: an electrostatic pump
- ❑ Currents
- ❑ Magnetic forces on point charges and currents



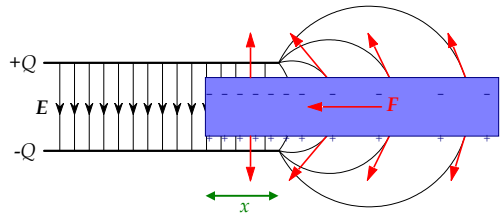
Force and work on dielectrics

A real parallel-plate capacitor – one with finite-size plates – has an electric field that extends past its boundaries. The field past the edges is called the fringing field.



Force and work on dielectrics (continued)

A dielectric slab inserted part-way into the capacitor is polarized by the field, and the bound charge experiences forces exerted by the fringing fields that push the dielectric further into the capacitor. What is the force?



Force and work on dielectrics (continued)

There are two ways we could go about the calculation:

- Calculate the fringing electric *displacement*, $\boldsymbol{D}$, by solving Laplace's equation for the boundary conditions presented by the capacitor plates. (If the plates are circles or squares, this would be straightforwardly solved by separation of variables.) Then put the dielectric slab back, and calculate the field and bound charge distribution. Then calculate the total force on the bound charges.

  Believe it or not, you now know how to do it this way, and I have half a mind to make this a homework problem. But I won't, because there's an easier way to do it that ignores the fringing field completely, even though that's what does the work.

Force and work on dielectrics (continued)

- We know the potential energy (work done by an external agent) for empty and dielectric-filled capacitors, so calculate W and use $\boldsymbol{F} = -\nabla W$ to compute the force.
  Let's begin by assuming that the charge on the capacitor is constant, and that the plate dimensions are $a \times \ell$, with separation d. Then

$$W = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C} \quad , \; Q \text{ constant.}$$

  Break the capacitor into two parts, one with dielectric and one without. These two capacitances are in parallel (have the same potential difference), so

$$C = C_{\text{dielectric}} + C_{\text{empty}} = \frac{\varepsilon a x}{4\pi d} + \frac{a(\ell - x)}{4\pi d}$$

Force and work on dielectrics (continued)

Then the force is

$$\boldsymbol{F} = -\nabla W = -\hat{\boldsymbol{x}}\frac{d}{dx}\left(\frac{1}{2}\frac{Q^2}{C}\right) = \hat{\boldsymbol{x}}\frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx}$$

$$= \hat{\boldsymbol{x}}\frac{1}{2}\frac{Q^2}{C^2}\left(\frac{\varepsilon a}{4\pi d} - \frac{a}{4\pi d}\right) = \hat{\boldsymbol{x}}\frac{1}{2}\frac{Q^2 a}{C^2 d}\frac{\varepsilon - 1}{4\pi} = \hat{\boldsymbol{x}}\frac{1}{2}\frac{V^2 a}{d}\chi_e \quad .$$

Sure enough, the dielectric is pulled into the capacitor. Here the charge is constant as this happens, so the potential difference V between the plates is less than is was when the capacitor was empty.

If the potential were fixed in the problem – if, say, a battery were present, then

Force and work on dielectrics (continued)

$$dW = F_{us}dx + VdQ = -Fdx + VdQ$$

$$F_x = -\frac{dW}{dx} + V\frac{dQ}{dx} = -\frac{1}{2}V^2\frac{dC}{dx} + V^2\frac{dC}{dx}$$

$$= \frac{1}{2}V^2\frac{dC}{dx} \qquad \text{Same as before...}$$

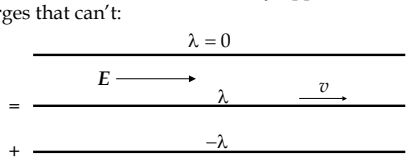
$$\boldsymbol{F} = \hat{\boldsymbol{x}}\frac{1}{2}\frac{V^2 a}{d}\chi_e \quad .$$

This time, the voltage is constant and the charge on the capacitor plates increases as the dielectric is pulled in. But either way, the dielectric is pulled in.

You will explore this some more on the homework in the classic "capacitor oil pump" problem, 4.28.

Currents

Consider a very long conducting wire, part of which is shown below. It is neutral overall but contains charges that can move under the influence of an externally-applied electric field, and charges that can't:



The motion of the $+\lambda$ distribution is a steady **current:**

$$\boxed{I = \lambda v}$$

Units: esu/sec in cgs, coul/sec = amp in MKS.

Currents (continued)

During a time Δt a charge $\lambda v \Delta t$ passes a given point on the wire; I is the rate at which charge passes that point.

- Though in general $\boldsymbol{v}$ is of course a vector, we don't usually write I as a vector, because we usually run currents through wires (one dimension).
- It is, however, useful to think of the surface and volume analogues of this line-charge current as vectors, since they will come into any discussion of currents *within* a wire.

Suppose we have a conducting sheet, with surface density σ in mobile charges and $-\sigma$ in fixed charges. Then the surface current density is

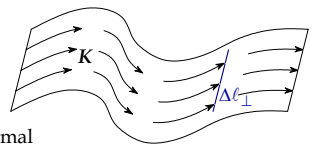
$$\boxed{\boldsymbol{K} = \sigma \boldsymbol{v}}$$

Units: esu/(cm sec) in cgs, amp/m in MKS.

Currents (continued)

The total current from the sheet is

$$\boldsymbol{I} = \int \boldsymbol{K} d\ell_\perp \quad \Leftrightarrow \quad \boldsymbol{K} = \frac{d\boldsymbol{I}}{d\ell_\perp}$$



where $d\ell_\perp$ is an infinitesimal length element perpendicular to the direction the current flows.

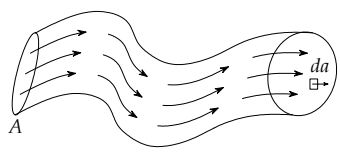
And, suppose you have a conducting volume, with charge density ρ of mobile charges and $-\rho$ of fixed charges. Then we get to define the most useful of current densities:

Currents (continued)

$$\boldsymbol{J} = \rho \boldsymbol{v}$$

$$\boldsymbol{I} = \int \boldsymbol{J} \cdot d\boldsymbol{a}$$

$$\boldsymbol{J} = \frac{d\boldsymbol{I}}{da_\perp}$$



The units of current density $\boldsymbol{J}$ are esu cm^2 sec^{-1} or amp m^{-2}. Integrating over the whole surface area of a conductor, one gets the current flowing out of the volume it encloses:

$$I = \oint_A \boldsymbol{J} \cdot d\boldsymbol{a} = \int_{\mathcal{V}} \nabla \cdot \boldsymbol{J} d\tau$$

Currents (continued)

Because electric charge is conserved, the only way for a current to flow out of a volume is for the charge inside it to decrease:

$$\oint_A \boldsymbol{J} \cdot d\boldsymbol{a} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \rho d\tau = -\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\tau \quad .$$

We can equate the integrands of the last two volume integrals to produce

$$\boxed{\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t} \quad .}$$

Continuity equation

Magnetic force on point charges

The reason we consider currents, of course, is that they exert forces on each other that cannot be explained by electrostatics: parallel currents attract each other, and opposite currents repel each other, for a force that is perpendicular to $\boldsymbol{v}$. As you are already aware, this is because of a new force that has nothing obvious (at the moment) to do with Coulomb's law, the **magnetic force**, and an associated **magnetic field, $\boldsymbol{B}$**. This force is empirically determined for a point charge to be

$$\boldsymbol{F}_{\text{mag}} = q\frac{\boldsymbol{v}}{c}\times\boldsymbol{B} \qquad \left[q\boldsymbol{v}\times\boldsymbol{B} \text{ in MKS units}\right] \qquad c = \text{speed of light}$$

$$\boxed{\boldsymbol{F} = \boldsymbol{F}_{\text{elec}} + \boldsymbol{F}_{\text{mag}} = q\boldsymbol{E} + q\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}} \qquad \text{Lorentz force law}$$

Magnetic units

$\boldsymbol{F} = q\boldsymbol{E} + q\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}$ In cgs, $\boldsymbol{E}$ and $\boldsymbol{B}$ have the same units, usually expressed as statvolt/cm for $\boldsymbol{E}$ or gauss (= statvolt/cm) for $\boldsymbol{B}$.

$\boldsymbol{F} = q\boldsymbol{E} + q\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}$ In MKS, $\boldsymbol{E}$ and $\boldsymbol{B}$ have different units:

$[\boldsymbol{E}]$ = V/m or Nt/coul

$[\boldsymbol{B}]$ = Tesla = T = Nt sec coul^{-1} m^{-1}

$= 10^4$ gauss .

Notes about B

- The electrostatic force can do work:

$$W = -\int_a^b \boldsymbol{F}\cdot d\boldsymbol{l} = qV(b) - qV(a) \quad ,$$

but the magnetic force cannot, because it's always perpendicular to the motion of the charges:

$$W = -\int_a^b \boldsymbol{F}\cdot d\boldsymbol{l} = -\frac{q}{c}\int_a^b (\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{v}dt = 0 \quad .$$

- The magnetic force on a current is

$$\boldsymbol{F}_{\text{mag}} = \int dq\frac{\boldsymbol{v}}{c}\times\boldsymbol{B} = \int \lambda d\ell\frac{\boldsymbol{v}}{c}\times\boldsymbol{B} = \int d\ell\frac{\boldsymbol{I}}{c}\times\boldsymbol{B} = \frac{I}{c}\int d\boldsymbol{l}\times\boldsymbol{B} \quad .$$

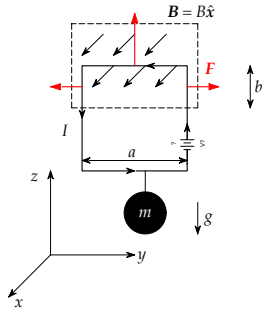
($d\boldsymbol{l}$ points along the direction the current flows.)

Force on a current

Example. A rectangular circuit, lying in the y-z plane, carries a current I and lies partly in a region where there is a constant magnetic field in the x direction. A weight, mass m, hangs from the circuit.

a. What is I, given m?

b. What happens if I is made larger than the value found in a?



Force on a current (continued)

a. $$\boldsymbol{F}_{\text{mag}} = \frac{I}{c}\int d\boldsymbol{l}\times\boldsymbol{B} = \frac{I}{c}\left(\int_0^b \hat{\boldsymbol{z}}\times\hat{\boldsymbol{x}}Bdz + \int_0^a -\hat{\boldsymbol{y}}\times\hat{\boldsymbol{x}}Bdy + \int_b^0 \hat{\boldsymbol{z}}\times\hat{\boldsymbol{x}}Bdz\right)$$

$$= \frac{IaB}{c}\hat{\boldsymbol{z}} = m\boldsymbol{g} \quad ,$$

so $$I = \frac{mgc}{aB} \quad .$$

b. The loop will rise until its lower edge enters the field region.
(Don't get the impression from this that the magnetic field is doing work to lift the weight; the work is done by the battery.)

Another note on *B*

- None of this tells us so far where ***B*** comes from. As you know, it is generated by currents, in a manner described by the Biot-Savart law, but we'll talk about that next time.