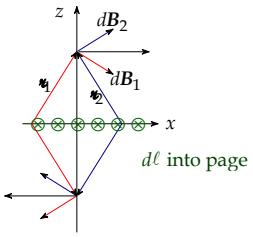
Today in Physics 217: magnetic vector potential

- Potentials, and the magnetic vector potential ***A***
- Arbitrariness of the potential: the Coulomb gauge
- ***A*** and the Biot-Savart law
- What is ***A***?
- Magnetic boundary conditions



Potentials

Recall the Helmholtz theorem (lecture, 16 September): any vector function ***F*** can be expressed as

$$\boldsymbol{F} = -\nabla U + \nabla \times \boldsymbol{W}$$

where

$$U(\boldsymbol{r}) = \frac{1}{4\pi}\int_V \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} D(\boldsymbol{r}')\,d\tau'$$

$$W(\boldsymbol{r}) = \frac{1}{4\pi}\int_V \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{C}(\boldsymbol{r}')\,d\tau'$$

Scalar and vector potential

and under the assumptions that, as $r \to \infty$,

$$\boldsymbol{F} \to 0, \quad r^2 \nabla \cdot \boldsymbol{F} \to 0, \quad \text{and } r^2 \nabla \times \boldsymbol{F} \to 0 \quad .$$

Potentials (continued)

We have also discussed two immediate consequences of the Helmholtz theorem :

- **Irrotational fields.** If a vector function is such that $\boldsymbol{F} = -\nabla U$, then all of the following are true:

  $\nabla \times \boldsymbol{F} = 0.$

  $\int_a^b \boldsymbol{F} \cdot d\boldsymbol{l}$ is independent of path, given ***a*** and ***b***.

  $\oint \boldsymbol{F} \cdot d\boldsymbol{l} = 0$ (thus "irrotational").

  In electrostatics, the electric field ***E*** is irrotational, and is the gradient of the (scalar) potential V.

Potentials (continued)

- **Solenoidal fields**. If a vector function is such that $\boldsymbol{F} = \nabla \times \boldsymbol{W}$, then all of the following are true:

  $\nabla \cdot \boldsymbol{F} = 0.$

  $\int_S \boldsymbol{F} \cdot d\boldsymbol{a}$ is independent of surface, given the boundary $\boldsymbol{C}$.

  $\oint \boldsymbol{F} \cdot d\boldsymbol{a} = 0.$

  In magnetostatics, the magnetic field $\boldsymbol{B}$ is solenoidal $(\nabla \cdot \boldsymbol{B} = 0)$, and is the curl of the **magnetic vector potential**:

  $$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad .$$

Potentials (continued)

Recall also that although $\boldsymbol{F}$ is a unique solution to the differential equations, the potentials U and $\boldsymbol{W}$ are not, just because the field depends upon the gradient and curl of these potentials, and:

- if a constant is added to U, the result has the same gradient, because the gradient of a constant is zero.
- Similarly, if a gradient is added to $\boldsymbol{W}$, the result has the same curl, because the curl of a gradient is zero.

Thus we can add an arbitrary constant to a scalar potential, and the gradient of an arbitrary scalar function to a vector potential, without changing the physics. To make special choices of these "offsets" can make certain problems easier to set up. A set of such choices is called a **gauge.**

Arbitrariness of A: the Coulomb gauge

$\boldsymbol{A}$ is arbitrary in the sense that one can add a gradient to it without changing $\boldsymbol{B}$, because the curl of any gradient is zero:

$$\nabla \times (\boldsymbol{A} + \nabla\lambda) = \nabla \times \boldsymbol{A} + \nabla \times \nabla\lambda = \nabla \times \boldsymbol{A} \quad .$$

As was the case for V, there is a *conventional* reference point, corresponding to our conventional choice of $V(\boldsymbol{r}) \to 0$ as $r \to \infty$. The convention comes from a consideration of Ampère's Law:

$$\nabla \times \boldsymbol{B} = \nabla \times \nabla \times \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$$

$$= \frac{4\pi}{c} \boldsymbol{J} \quad .$$

If $\nabla \cdot \boldsymbol{A}$ were zero, then Ampère's law would yield an expression resembling the Poisson equation. In electrodynamics this choice is called the **Coulomb gauge**.

Arbitrariness of *A*: the Coulomb gauge (continued)

$$\nabla \cdot \boldsymbol{A} = 0 \quad \Rightarrow \quad \nabla^2 \boldsymbol{A} = -\frac{4\pi}{c} \boldsymbol{J} \quad ,$$

which might be handy, and also makes magnetostatics look even more like electrostatics.

- But is there always a scalar function λ such that its gradient can be added to $\boldsymbol{A}_0$ to make a new vector potential with zero divergence? Yes. Here's why. Let

$$\boldsymbol{A} = \boldsymbol{A}_0 + \nabla\lambda \quad ,$$

where $\nabla\lambda$ is chosen such that

$$\nabla \cdot \boldsymbol{A} = 0 = \nabla \cdot \boldsymbol{A}_0 + \nabla^2\lambda \quad \Rightarrow \quad \nabla^2\lambda = -\nabla \cdot \boldsymbol{A}_0$$

Arbitrariness of *A*: the Coulomb gauge (continued)

- But this is just a Poisson equation, and we know it has a solution, so the appropriate function λ exists. We even know what it is, because it is governed by the same relation as the electrostatic scalar potential, for which the general solution is known by Coulomb's law:

$$\lambda = \frac{1}{4\pi} \int \frac{\nabla' \cdot \boldsymbol{A}_0(\boldsymbol{r}')}{\mathscr{r}} d\tau' \quad .$$

(That turns out not to be why they call it the Coulomb gauge, though.)

A and the Biot-Savart law

We showed on Wednesday, using the Biot-Savart law, that

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{1}{c} \nabla \times \int_{\mathcal{V}} \frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}} d\tau' \quad .$$

Apparently,

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{c} \int_{\mathcal{V}} \frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}} d\tau' \qquad \left(\text{cf. } V(\boldsymbol{r}) = \int \frac{\rho(\boldsymbol{r}')}{\mathscr{r}} d\tau' \right) \quad .$$

This turns out already to be divergenceless:

$$\nabla \cdot A(\boldsymbol{r}) = \frac{1}{c} \int_{\mathcal{V}} \nabla \cdot \left(\frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}} \right) d\tau' \qquad \text{Apply Product Rule \#5:}$$

A and the Biot-Savart law (continued)

$$\nabla\cdot\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{c}\int_{\mathcal{V}}\left[\frac{1}{\mathscr{r}}\nabla\cdot\boldsymbol{J}(\boldsymbol{r}') + \boldsymbol{J}(\boldsymbol{r}')\cdot\nabla\left(\frac{1}{\mathscr{r}}\right)\right]d\tau' \qquad (\nabla\cdot\boldsymbol{J}=0)$$

$\boldsymbol{J}(\boldsymbol{r}')$ doesn't depend on $\boldsymbol{r}$

$$= -\frac{1}{c}\int_{\mathcal{V}}\boldsymbol{J}(\boldsymbol{r}')\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)d\tau'$$

because $\nabla\left(\frac{1}{\mathscr{r}}\right) = -\nabla'\left(\frac{1}{\mathscr{r}}\right)$

$$= -\frac{1}{c}\int_{\mathcal{V}}\left[\nabla'\cdot\left(\frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}}\right) - \frac{1}{\mathscr{r}}\nabla'\cdot\boldsymbol{J}(\boldsymbol{r}')\right]d\tau' \qquad (\nabla'\cdot\boldsymbol{J}=0)$$

magnetostatics!

$$= -\frac{1}{c}\oint_{\mathcal{S}}\frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}}\cdot d\boldsymbol{a}'$$

divergence theorem

$$= 0$$

by definition, $\boldsymbol{J} = 0$ everywhere on $\mathcal{S}$.

What is the magnetic vector potential?

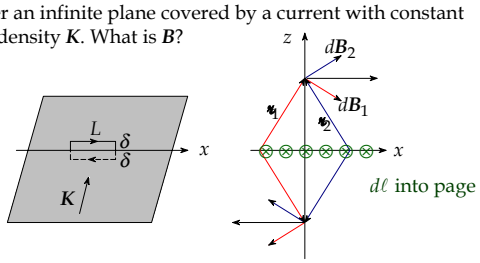
- Unlike V, which we think of as work per unit charge, there's no obvious mechanical interpretation of $\boldsymbol{A}$.
- Momentum per unit charge (times c) comes closest. For instance, the canonical momentum of a charged particle in an electromagnetic field is

$$\boldsymbol{p}_{\text{canonical}} = \boldsymbol{p} - \frac{q}{c}\boldsymbol{A} \quad ,$$

  and as such it appears frequently in the equations of quantum mechanics.
- Also unlike V, it's a vector. Since $\boldsymbol{B}$ is solenoidal it can't have a scalar potential. Thus $\boldsymbol{A}$ is much less useful in magnetostatic calculations than V is in electrostatics.

Magnetic boundary conditions

Consider an infinite plane covered by a current with constant surface density $\boldsymbol{K}$. What is $\boldsymbol{B}$?



Solve with Ampère's law with the loop as shown, as the vertical components cancel and the horizontal ones add.

Magnetic boundary conditions (continued)

$$\oint \boldsymbol{B}\cdot d\ell = \frac{4\pi}{c} I_{\text{enclosed}}$$

$$BL + 0 + BL + 0 = \frac{4\pi}{c} KL$$

$$\Rightarrow \boldsymbol{B} = \frac{2\pi K}{c}\hat{\boldsymbol{x}} \quad (z>0), \quad -\frac{2\pi K}{c}\hat{\boldsymbol{x}} \quad (z<0).$$

$$\Rightarrow \boldsymbol{B}_{\text{above}} - \boldsymbol{B}_{\text{below}} = \frac{4\pi}{c}\boldsymbol{K}\times\hat{\boldsymbol{z}} \quad .$$

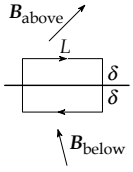
This problem is the paradigm for obtaining the boundary conditions for magnetostatic boundary-value problems:

Magnetic boundary conditions (continued)

If $\delta \to 0$, Ampère's law becomes

$$\left(B_{\parallel,\text{ above}} - B_{\parallel,\text{ below}}\right)L = \frac{4\pi}{c} KL \quad ,$$

as the contributions from the perpendicular components of $\boldsymbol{B}$ vanish as the sides of the loop approach zero size.



Thus

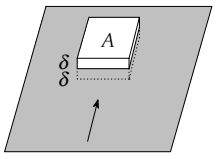
$$B_{\parallel,\text{ above}} - B_{\parallel,\text{ below}} = \frac{4\pi}{c} K \quad .$$

For the components perpendicular to the surface, use $\nabla\cdot\boldsymbol{B} = 0$, with a Gaussian pillbox that has very short sides:

Magnetic boundary conditions (continued)

$$\oint B\cdot da = 0$$

$$B_{\perp,\text{ above}} A - B_{\perp,\text{ below}} A =$$



Combine with result for parallel components:

$$\boldsymbol{B}_{\text{above}} - \boldsymbol{B}_{\text{below}} = \frac{4\pi}{c}\boldsymbol{K}\times\hat{\boldsymbol{n}} \quad .$$

(cf. $\boldsymbol{E}_{\text{above}} - \boldsymbol{E}_{\text{below}} = 4\pi\sigma\hat{\boldsymbol{n}}$.)

Similarly, $\boldsymbol{A}_{\text{above}} - \boldsymbol{A}_{\text{below}} = 0$,

$$\frac{\partial}{\partial n}\boldsymbol{A}_{\text{above}} - \frac{\partial}{\partial n}\boldsymbol{A}_{\text{below}} = -\frac{4\pi}{c}\boldsymbol{K} \qquad \left(\frac{\partial}{\partial n} \equiv \hat{\boldsymbol{n}}\cdot\nabla\right).$$