Today in Physics 217: magnetic vector potential

- Potentials, and the magnetic vector potential $A$
- Arbitrariness of the potential: the Coulomb gauge
- $A$ and the Biot-Savart law
- What is $A$?
- Magnetic boundary conditions

Potentials

Recall the Helmholtz theorem (lecture, 16 September): any vector function $F$ can be expressed as

$$F = -\nabla U + \nabla \times W$$

where

$$U(r) = \frac{1}{4\pi} \int \frac{1}{|r-r'|} D(r')\,dr'$$

Scalar and vector potential

$$W(r) = \frac{1}{4\pi} \int \frac{1}{|r-r'|} C(r')\,dr'$$

and under the assumptions that, as $r \to \infty$,

$$F \to 0, \quad r^2 \nabla \cdot F \to 0, \quad and \quad r^2 \nabla \times F \to 0.$$

Potentials (continued)

We have also discussed two immediate consequences of the Helmholtz theorem:

- **Irrotational fields.** If a vector function is such that $F = -\nabla U$, then all of the following are true:
  - $\nabla \times F = 0$.
  - $\int_a^b F \cdot dl$ is independent of path, given $a$ and $b$.

In electrostatics, the electric field $E$ is irrotational, and is the gradient of the (scalar) potential $V$. 
Solenoidal fields. If a vector function is such that \( \mathbf{F} = \nabla \times \mathbf{W} \), then all of the following are true:

\[
\nabla \cdot \mathbf{F} = 0.
\]

\[
\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{a} \quad \text{is independent of surface, given the boundary \( \mathcal{C} \).}
\]

\[
\oint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{a} = 0.
\]

In magnetostatics, the magnetic field \( \mathbf{B} \) is solenoidal \((\nabla \cdot \mathbf{B} = 0)\), and is the curl of the magnetic vector potential:

\[
\mathbf{B} = \nabla \times \mathbf{A}.
\]

Recall also that although \( \mathbf{F} \) is a unique solution to the differential equations, the potentials \( U \) and \( W \) are not, just because the field depends upon the gradient and curl of these potentials, and:

- If a constant is added to \( U \), the result has the same gradient, because the gradient of a constant is zero.
- Similarly, if a gradient is added to \( W \), the result has the same curl, because the curl of a gradient is zero.

Thus we can add an arbitrary constant to a scalar potential, and the gradient of an arbitrary scalar function to a vector potential, without changing the physics. To make special choices of these “offsets” can make certain problems easier to set up. A set of such choices is called a gauge.

Arbitrariness of \( A \): the Coulomb gauge

\( A \) is arbitrary in the sense that one can add a gradient to it without changing \( \mathbf{B} \), because the curl of any gradient is zero:

\[
\nabla \times (A + \nabla \lambda) = \nabla \times A + \nabla \times \nabla \lambda = \nabla \times A.
\]

As was the case for \( V \), there is a conventional reference point, corresponding to our conventional choice of \( V(r) \rightarrow 0 \) as \( r \rightarrow \infty \). The convention comes from a consideration of Ampère’s Law:

\[
\nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 A
\]

\[
= \frac{4\pi}{c} \mathbf{J}.
\]

If \( \nabla \cdot \mathbf{A} \) were zero, then Ampère’s law would yield an expression resembling the Poisson equation. In electrodynamics this choice is called the Coulomb gauge.
Arbitrariness of \( A \): the Coulomb gauge (continued)

\[ \nabla \cdot A = 0 \Rightarrow \nabla^2 A = -\frac{4\pi}{c} j , \]

which might be handy, and also makes magnetostatics look even more like electrostatics.

\[ \text{But is there always a scalar function } \lambda \text{ such that its} \]
\[ \text{gradient can be added to } A_0 \text{ to make a new vector} \]
\[ \text{potential with zero divergence? Yes. Here's why. Let} \]
\[ A = A_0 + \nabla \lambda , \]
\[ \text{where } \nabla \lambda \text{ is chosen such that} \]
\[ \nabla \cdot A = 0 = \nabla \cdot A_0 + \nabla^2 \lambda \Rightarrow \nabla^2 \lambda = -\nabla \cdot A_0 \]

Arbitrariness of \( A \): the Coulomb gauge (continued)

\[ \text{But this is just a Poisson equation, and we know it has a} \]
\[ \text{solution, so the appropriate function } \lambda \text{ exists. We even} \]
\[ \text{know what it is, because it is governed by the same} \]
\[ \text{relation as the electrostatic scalar potential, for which the} \]
\[ \text{general solution is known by Coulomb's law:} \]
\[ \lambda = \frac{1}{4\pi} \int \left( \nabla \cdot A_0 (r)^2 \right) \text{dr} . \]

(That turns out not to be why they call it the Coulomb gauge, though.)

\[ A \text{ and the Biot-Savart law} \]

We showed on Wednesday, using the Biot-Savart law, that
\[ B(r) = \frac{1}{c} \nabla \times \int \frac{J(r')}{r} \text{dr} . \]

Apparently,
\[ A(r) = \frac{1}{c} \int \frac{J(r')}{r} \text{dr} . \]

(cf. \( V(r) = \int \frac{P(r')}{r} \text{dr} \).

This turns out already to be divergenceless:
\[ \nabla \cdot A(r) = \frac{1}{c} \int \nabla \cdot \left( \frac{J(r')}{r} \right) \text{dr} . \quad \text{Apply Product Rule #5} \]
A and the Biot-Savart law (continued)

\[ \nabla \cdot A(r) = \frac{1}{c} \int \frac{1}{\epsilon} \nabla \cdot \mathbf{J}(r') + \mathbf{J}(r') \cdot \nabla \left( \frac{1}{\epsilon} \right) \, dt' \]

\[ J(r') \text{ doesn't depend on } r \]

\[ = \frac{1}{c} \int \nabla \cdot \mathbf{J}(r') \, dt' \]

\[ = \frac{1}{\epsilon} \int \mathbf{V} \cdot \mathbf{J}(r') \, dt' \]

\[ = -\frac{1}{\epsilon} \int \mathbf{V} \cdot \mathbf{J}(r') \, dt' \]

because \( \mathbf{V} \left( \frac{1}{\epsilon} \right) = -\mathbf{V} \left( \frac{1}{\epsilon} \right) \)

\[ = 0 \] magnetostatics!

\[ = -\frac{1}{c} \int \mathbf{J}(r') \cdot \mathbf{a}' \]

\[ = 0 \] divergence theorem

\[ = 0 \] by definition, \( J = 0 \) everywhere on \( S \).

What is the magnetic vector potential?

- Unlike \( V \), which we think of as work per unit charge, there's no obvious mechanical interpretation of \( A \).
- Momentum per unit charge (times \( c \)) comes closest. For instance, the canonical momentum of a charged particle in an electromagnetic field is

\[ p_{\text{canonical}} = p - \frac{q}{c} A \]

and as such it appears frequently in the equations of quantum mechanics.
- Also unlike \( V \), it's a vector. Since \( B \) is solenoidal it can't have a scalar potential. Thus \( A \) is much less useful in magnetostatic calculations than \( V \) is in electrostatics.

Magnetic boundary conditions

Consider an infinite plane covered by a current with constant surface density \( K \). What is \( B \)?

\[ dB_2 = \nabla \times B_1 \]

Solve with Ampère's law with the loop as shown, as the vertical components cancel and the horizontal ones add.
Magnetic boundary conditions (continued)

\[ \oint B \cdot dl = \frac{4\pi}{c} I_{\text{enclosed}} \]

\[ BL + 0 + BL + 0 = \frac{4\pi}{c} K L \]

\[ \Rightarrow B = \frac{2\pi K}{c} \hat{k} \ (z > 0), \quad -\frac{2\pi K}{c} \hat{k} \ (z < 0). \]

\[ \Rightarrow B_{\text{above}} - B_{\text{below}} = \frac{4\pi}{c} K \times \hat{z}. \]

This problem is the paradigm for obtaining the boundary conditions for magnetostatic boundary-value problems:

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Magnetic boundary conditions (continued)

If \( \delta \to 0 \), Ampère’s law becomes

\[ (B_{\text{above}} - B_{\text{below}}) \cdot L = \frac{4\pi}{c} K L, \]

as the contributions from the perpendicular components of \( B \) vanish as the sides of the loop approach zero size.

Thus

\[ B_{\text{above}} - B_{\text{below}} = \frac{4\pi}{c} K. \]

For the components perpendicular to the surface, use

\[ \nabla \cdot B = 0, \quad \text{with a Gaussian pillbox that has very short sides:} \]

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Magnetic boundary conditions (continued)

Combine with result for parallel components:

\[ \oint B \cdot da = \frac{4\pi}{c} K \cdot \hat{n}. \]

\( (\text{cf.} \ E_{\text{above}} - E_{\text{below}} = \frac{4\pi}{c} \hat{n} \cdot \hat{n}) \)

Similarly, \( A_{\text{above}} - A_{\text{below}} = 0 \),

\[ \frac{\partial}{\partial n} A_{\text{above}} - \frac{\partial}{\partial n} A_{\text{below}} = \frac{4\pi}{c} K \left( \frac{\partial}{\partial n} = \hat{n} \cdot \hat{V} \right). \]