Today in Physics 217: magnetism in matter

- Magnetism
- Magnetization and bound currents
- Ampère’s Law and magnets
- Linear magnetic media
- Cautions about $H$
- Example: $H$ and Ampère’s Law

Magnetism

There are three forms of magnetism in matter, all having to do with spins of atomic electrons.

- Electrons have spin, and spinning charges have magnetic dipole moments that can be aligned by external magnetic fields.
- However, the spin of electrons, and the pairings of electrons within atoms, are manifestations of quantum-mechanical effects, so magnetism is not really understandable simply from our present classical viewpoint.

Anyway, here are the three kinds of magnetism:

Magnetism (continued)

- Ferromagnetism
  The layman’s magnetism, this is the only strong form of magnetism.
  - It comes from aligned spins of unpaired $d$ electrons in transition metals, especially nickel, iron and cobalt (hence the name), and rare earths like samarium.

- Paramagnetism
  Analogous to electric polarization, and much weaker than ferromagnetism; can almost understand classically.
  - Produced by interaction of $B$ with dipole moments of unpaired $s$ or $p$ electrons. Aluminum, for example, has one $p$ electron in its valence shell, and is paramagnetic. So is molecular oxygen, which has two unpaired spins among its bonding electrons.
Magnetism (continued)

- **Diamagnetism**
  - Comes from the interaction of $B$ with induced magnetic dipoles in atoms; also much weaker than ferromagnetism.
  - Contrary to the way electric polarization works, diamagnetism is characterized by magnetization in the direction opposite that of the applied field.
  - Seen best in materials in which all the electrons are paired off (Ne, N₂, ...).

Whatever the type, we can define a magnetization, in analogy with the electric polarization $P$:

$$M = \frac{\text{magnetic dipole moment}}{\text{unit volume}}$$

Magnetization and bound currents

Like $P$, which we interpret in terms of bound charge, $M$ can be interpreted in terms of bound currents, which we can characterize by consideration of the magnetic potential for a dipole.

$$A = \oint_S \mathbf{M} \cdot d\mathbf{r} = \oint_S \mathbf{M} \cdot d\mathbf{r}'$$

$$= \oint_S \mathbf{M} \times \left( \frac{1}{r^3} \right) d\mathbf{r}'$$

Use Product Rule #7:

$$= \oint_S \mathbf{M} \times \mathbf{r}'$$

Magnetization and bound currents (continued)

Now, for any vector function $\mathbf{v} = \mathbf{v}(r)$ and a constant vector $\mathbf{C}$, we can write the divergence theorem as

$$\int_V \nabla \cdot (\mathbf{v} \times \mathbf{C}) = \oint_{\partial V} (\mathbf{v} \times \mathbf{C}) \cdot d\mathbf{a}$$

$$\int_V (\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{C}) = \oint_{\partial V} \mathbf{v} \times (\mathbf{C} \cdot d\mathbf{a})$$

Product Rule #6

$$C \cdot \int_V \nabla \cdot \mathbf{v} dt = \oint_{\partial V} \mathbf{v} \times (C \cdot d\mathbf{a})$$

$C$ is constant

$$\mathbf{C} \cdot \int_V \nabla \cdot \mathbf{v} dt = \oint_{\partial V} \mathbf{v} \times (\mathbf{C} \cdot d\mathbf{a})$$

$C$ is constant

Triple product rule #1

$$\mathbf{C} \cdot \int_V \nabla \cdot \mathbf{v} dt = \oint_{\partial V} \mathbf{v} \times (\mathbf{C} \cdot d\mathbf{a})$$

Thus $A = \oint_S \mathbf{M} \times (d\mathbf{r} + \frac{1}{S} \mathbf{M} \times d\mathbf{a})$. 
Magnetization and bound currents (continued)

Thus
\[ A = \frac{1}{c} \nabla \times M \, dt' + \frac{1}{S} \int \mathbf{M} \times d\mathbf{a}' \]
\[ = \frac{1}{c} \int \mathbf{J}_b \, dt' + \frac{1}{c} \int \mathbf{K}_b \, d\mathbf{a}' , \]
where we have defined the bound current densities:
\[ \mathbf{J}_b (r') = c \nabla \times \mathbf{M} (r'), \quad \text{(cf. } P_0 = -\nabla \cdot \mathbf{P}, \text{)} \]
\[ \mathbf{K}_b (r') = c \mathbf{M} \times \hat{n}_S . \]

[In MKS, \( \mathbf{J}_b (r') = \nabla \times \mathbf{M} (r'), \quad \mathbf{K}_b (r') = \mathbf{M} \times \hat{n}_S , \]
\[ A = \frac{\mu_0}{4\pi} \int \mathbf{J}_b \, dt' + \frac{\mu_0}{4\pi} \int \mathbf{K}_b \, d\mathbf{a}' . \]

Ampère’s Law revisited

With bound currents we can do to Ampère’s Law about what we did to Gauss’s law before:
\[ \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} = \frac{4\pi}{c} (\mathbf{J}_{\text{free}} + \mathbf{J}_b) = \frac{4\pi}{c} \left( \mathbf{J}_f + c \nabla \times \mathbf{M} \right) \]
\[ \nabla \times (\mathbf{B} - 4\pi \mathbf{M}) = \frac{4\pi}{c} \mathbf{J}_f \]
\[ \Rightarrow \quad \nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J}_f , \quad \mathbf{H} = \mathbf{B} - 4\pi \mathbf{M} \quad \text{(cf. } \mathbf{D} = \varepsilon \mathbf{E}) \]

Or, in MKS, \( \nabla \times \mathbf{H} = \mathbf{J}_f , \quad \mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} . \)

Linear magnetic media

As before, when the definition of \( \mathbf{D} \) led to that of the electric susceptibility \( \chi_e \) and the dielectric constant \( \varepsilon \), we can define
\[ \mathbf{M} = \chi_m \mathbf{H} \]
That would be too sensible.

In these terms,
\[ \mathbf{B} = \mathbf{H} + 4\pi \mathbf{M} = \mathbf{H} (1 + 4\pi \chi_m) = \mu \mathbf{H} \quad \text{(cf. } \mathbf{D} = \varepsilon \mathbf{E}) \]
or, in MKS,
\[ \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu_0 (1 + \chi_m) \mathbf{H} = \mu \mathbf{H} . \]
\( \mu \) is called the relative permeability. Contrary to \( \varepsilon, \mu \) is the same in cgs and MKS, though \( \chi_m \), equally dimensionless, is not.
Linear magnetic media (continued)

The run of $\chi_m$ for the three forms of magnetism:

- **Ferromagnetism**: $\chi_m > 0$ and very large.
  For instance, pure, annealed iron has $\mu \approx 10000$.
- **Paramagnetism**: $\chi_m > 0$ but $\ll 1$.
  Typically, $\chi_m (\text{psp}) \approx 10^{-6}$.
  $\mu \approx 1$.

- **Diamagnetism**: $\chi_m < 0$, $|\chi_m| < 1$.
  Typically somewhat smaller than is typical in paramagnetism.

The consequence is that $\mu$ is so close to unity for anything that isn’t ferromagnetic, that one could easily avoid noticing magnetism in everyday life.

---

Linear magnetic media (continued)

Practical realization of the three different kinds of linear magnetic media, compared to electric polarization:

- **Capacitor and dielectric**
  - Force on dielectric

- **Solenoid and magnet**
  - Force: paramagnetic
  - diamagnetic
  - ferromagnetic

---

Cautions (?) about $H$

One must be careful about the use of $H$, for many of the same reasons that we are cautious about $D$. But $H$ does turn out to be somewhat more generally useful than $D$.

- **Reason**: measurability. We can easily measure potential differences and free currents (with voltmeters and ammeters), so these relate best to the fields $E$ and $H$, not $D$ and $B$.

- **Rule of thumb**: $H$ is useful whenever the situation is symmetrical enough to use Ampère’s Law: infinite cylinders, planes and solenoids, toroids.

Remember, though, $B$ is the fundamental field. There isn’t a Biot-Savart law or Lorentz force law for $H$. 
Example: \( H \) and Ampère’s Law

Example 6.3: An infinite solenoid carries a current \( I \), has \( n \) turns per unit length, and is filled with a magnetic material with relative permeability \( \mu \). What is the magnetic field \( B \) inside, and what are the bound currents associated with the material?

Because there are bound currents we can’t compute \( B \) directly, but we can compute \( H \).

Example: \( H \) and Ampère’s Law (continued)

Clearly the field should be zero outside, axial inside, and perpendicular to the two vertical sides of the Ampèrean loop.

Integrate over area, use Stokes’ theorem:

\[
\oint H \cdot dl = \frac{4\pi}{c} I_f, \text{enclosed}
\]

\[
H \cdot dl = H = \frac{4\pi}{c} n \ell \hat{z}, \quad B = \mu H = \frac{4\pi}{c} \mu n \ell \hat{z}.
\]

\[
K_b = \epsilon M \times \hat{n} = \epsilon \mu_m H \times \hat{s} = 4\pi \chi_m n \ell \hat{\phi},
\]

\[
J_b = \epsilon \nabla \times M = \nabla \times \chi_m H = 0.
\]

In general, the bound current will reside on the surface (unless free current flows in the medium), and will follow the free current.

Example: \( H \) and Ampère’s Law (continued)

The same thing, in MKS units:

Integrate over area, use Stokes’ theorem:

\[
\oint H \cdot dl = I_f, \text{enclosed}
\]

\[
H \cdot dl = n l \hat{z}, \quad B = \mu n l \hat{z}.
\]

Bound charges:

\[
K_b = M \times \hat{n} = \chi_m H \times \hat{s} = \chi_m n \ell \hat{\phi},
\]

\[
J_b = \nabla \times M = \nabla \times \chi_m H = 0.
\]