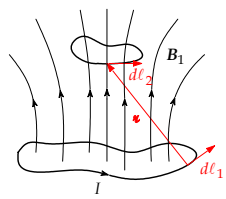
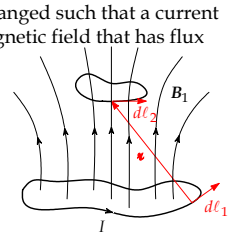
Today in Physics 217: inductance

- Mutual inductance
- How mutual inductance can help with magnetic flux calculations: examples of coupled solenoids and coupled circular loops.
- Self inductance and back EMF
- Units
- Calculation of inductance



Mutual inductance

Consider two *coupled* wire loops, arranged such that a current through one of them produces a magnetic field that has flux through the other one. For instance,



$$\Phi_{B2} = \int \boldsymbol{B}_1 \cdot d\boldsymbol{a}_2 = \int (\nabla \times \boldsymbol{A}_1) \cdot d\boldsymbol{a}_2$$

$$= \oint \boldsymbol{A}_1 \cdot d\ell_2 \quad .$$

But $\boldsymbol{A}_1 = \frac{I_1}{c} \oint \frac{d\ell_1}{\mathscr{r}}$, so

$$\Phi_{B2} = \frac{I_1}{c} \oint \frac{d\ell_2 \cdot d\ell_2}{\mathscr{r}} \propto I_1 \quad ;$$

the proportionality factor is a constant that depends only upon the geometry of the loops.

Mutual inductance (continued)

Now suppose we exchange the roles of the loops, and run a current through loop #2. We'd get the same result with just the 1s and 2s switched:

$$\Phi_{B1} = \frac{I_2}{c} \oint \frac{d\ell_2 \cdot d\ell_2}{\mathscr{r}} \propto I_2 \quad .$$

The proportionality factor is the same in the two cases. Define it as follows:

$$\Phi_{B1} = cMI_2 \quad , \quad \Phi_{B2} = cMI_1 \quad , \text{ where}$$

$$M = \frac{1}{c^2} \oint \frac{d\ell_2 \cdot d\ell_2}{\mathscr{r}} \quad \left[= \frac{\mu_0}{4\pi} \oint \frac{d\ell_2 \cdot d\ell_2}{\mathscr{r}} \text{ in MKS} \right]$$

is called the **mutual inductance** of the loops.

How mutual inductance can help with magnetic flux calculations

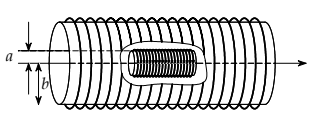
This simple exercise has a profound conclusion: no matter what the shape of two loops, **the flux through loop 2, resulting from current *I* in loop 1, is exactly the same as the flux through loop 1 due to current *I* in loop 2**.

- This occasionally is useful in calculations of flux and related quantities: switching the currents and fluxes for the calculation, and then switching back, can save a lot of trouble.
- Consider **Example 7.10**: a short solenoid (length ℓ, radius a, n_1 turns per unit length) and a very long one (b, n_2) are coaxial. Current I flows in the short one. What is the flux through the long one?

Coupled solenoids

This would be difficult if calculated the way it's stated, but easy if one instead runs the same current through the outer coil and calculates the flux through the inner:

$$\Phi_B = \frac{4\pi}{c} n_2 I \times n_1 \ell \pi a^2 \quad .$$



Coupled circular loops

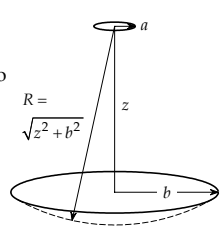
(Problem 7.20).

A small loop of wire (radius a) lies a distance z above the center of a large loop (radius b). The planes of the two loops are parallel, and perpendicular to the common axis.

(a) Suppose current I flows in the big loop. Find the flux in the little loop.

(b) Suppose current I flows in the little loop. Find the flux through the big loop.

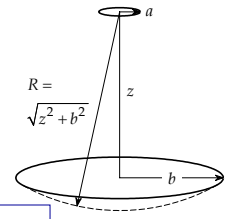
(c) What is the mutual inductance?



Coupled circular loops (continued)

(a) We saw long time ago (Example 5.6, lecture notes for 4 November 2002) that

$$\boldsymbol{B} = \frac{2\pi}{c}\frac{I_b b^2}{\left(z^2+b^2\right)^{3/2}}\hat{\boldsymbol{z}} \quad ,$$



so if the loop is small enough to consider $\boldsymbol{B}$ to be constant over its extent,

$$\Phi_{Ba} = \int \boldsymbol{B}_b \cdot d\boldsymbol{a}_a \cong B_b A_a = \boxed{\frac{2\pi^2}{c}\frac{b^2a^2}{\left(z^2+b^2\right)^{3/2}} I_b} \quad .$$

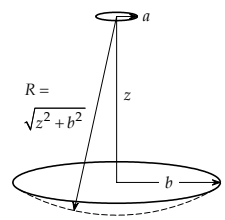
Coupled circular loops (continued)

(b) Supposing that the small loop is small enough to treat as a dipole,

$$\boldsymbol{B}_a = \frac{2m\cos\theta}{r^3}\hat{\boldsymbol{r}} + \frac{m\sin\theta}{r^3}\hat{\boldsymbol{\theta}} \quad ,$$

where $m = \frac{1}{c}\pi a^2 I_a$.

Then $\Phi_{Bb} = \int \boldsymbol{B}_a \cdot d\boldsymbol{a}_b$.



We can compute the flux over any surface that has the loop with radius b as its boundary (Problem 7.9, on this week's homework); looks easiest if we take it over the sphere with radius R centered in the center of the small loop:

Coupled circular loops (continued)

$$\Phi_{Bb} = \int \boldsymbol{B}_a \cdot d\boldsymbol{a}_b = \int \frac{2m\cos\theta}{R^3} R^2 \sin\theta\, d\theta\, d\phi$$

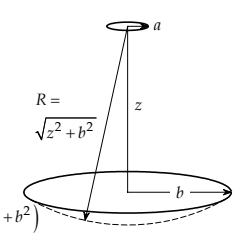
$$= \frac{2\pi m}{R}\int_0^{\theta_0} 2\cos\theta\,\sin\theta\, d\theta = \frac{\pi m}{R}\int_0^{2\theta_0} \sin u\, du$$

$$= -\frac{\pi m}{R}\cos u\Big|_0^{2\theta_0} = \frac{\pi m}{R} - \frac{\pi m}{R}\cos 2\theta_0$$

$$= \frac{\pi m}{R} - \frac{\pi m}{R}\left(\cos^2\theta_0 - \sin^2\theta_0\right)$$

$$= \frac{\pi m}{R} - \frac{\pi m}{R}\left(\frac{z^2}{R^2} - \frac{b^2}{R^2}\right) = \frac{\pi m}{R^3}\left(z^2 + a^2 - z^2 + b^2\right)$$

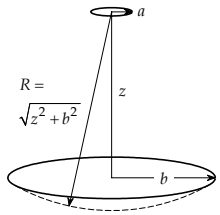
$$= \frac{2\pi m b^2}{\left(z^2+b^2\right)^{3/2}} = \boxed{\frac{2\pi^2}{c}\frac{a^2b^2}{\left(z^2+b^2\right)^{3/2}} I_a} \quad .$$



Coupled circular loops (continued)

(c) Thus

$$M = \frac{\Phi_{Ba}}{cI_b} = \frac{\Phi_{Bb}}{cI_a} = \boxed{\frac{2\pi^2}{c^2}\frac{b^2a^2}{\left(z^2+b^2\right)^{3/2}}} \quad .$$



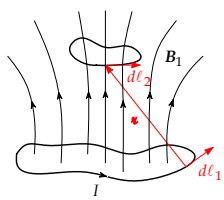
Note that it's much easier to get the fluxes -- and anything related to them, such as induced currents and fields -- by means of part a than part b.

Replace $1/c^2$ with $\mu_0/4\pi$ for the MKS answer.

Transformers

Back to our original loops. Suppose that the current in loop 1 varies with time. Then there is an EMF induced in loop 2:

$$\mathcal{E}_2 = -\frac{1}{c}\frac{d\Phi_{B2}}{dt} = -M\frac{dI_1}{dt} \quad .$$



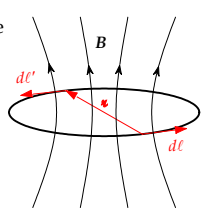
Thus

$$I_2 = \frac{\mathcal{E}_2}{R} = -\frac{M}{R}\frac{dI_1}{dt} \quad ;$$

a time-varying current in one loop ends up inducing a time-varying current in the other, with relative size M.

Self inductance and back EMF

What about one loop by itself? When one tries to change I, one changes the flux of the loop's own $\boldsymbol{B}$ through itself, and thus produces an opposing EMF.



$$\Phi_B = \int \boldsymbol{B}\cdot d\boldsymbol{a} = \oint \boldsymbol{A}\cdot d\ell$$

$$= \frac{I}{c}\oiint \frac{d\ell'\cdot d\ell}{\mathscr{r}} \equiv cLI \quad \text{, where}$$

$$L = \frac{1}{c^2}\oiint \frac{d\ell'\cdot d\ell}{\mathscr{r}}$$

$$\left[= \frac{\mu_0}{4\pi}\oiint \frac{d\ell'\cdot d\ell}{\mathscr{r}} \text{ in MKS}\right].$$

Self inductance

Self inductance and back EMF

This opposing EMF,

$$\mathcal{E} = -\frac{1}{c}\frac{d\Phi_B}{dt} = -L\frac{dI}{dt} \quad , \text{ Back EMF}$$

is a form of inertia in electrical circuits. If one tries to change a current, a back EMF in induced. We will see soon that in the "equations of motion" for AC circuits, L plays the role that m plays in classical-mechanical equations.
Work per unit charge done against a back EMF is just -$\mathcal{E}$:

$$P = \frac{dW}{dt} = -\mathcal{E}I = LI\frac{dI}{dt}$$

$$W = \int dW = L\int IdI = \frac{1}{2}LI^2 \quad .$$

Compare: $W = \frac{1}{2}CV^2$ for capacitors.

Units

$$[L] = \frac{[\mathcal{E}]}{\left[\frac{dI}{dt}\right]} = \frac{\left[\frac{Q}{r}\right]}{\left[\frac{Q/t}{t}\right]} = \text{ sec}^2 \text{ cm}^{-1} \text{ in cgs units;}$$

$$= \frac{\text{volt sec}}{\text{amp}} \equiv \text{ henry in MKS units.}$$

$$1 \text{ henry} \quad \Leftrightarrow \quad \frac{1}{9}\times 10^{-11} \text{sec}^2 \text{ cm}^{-1}$$

Common values in the lab:

$$L \approx 1 \text{ mH - } 1\ \mu\text{H}$$

$$\approx 10^{-15} - 10^{-18} \text{ sec}^2 \text{ cm}^{-1} \quad .$$

Calculation of inductance

We have the two formulas,

$$M = \frac{1}{c^2}\oint \frac{d\ell_2 \cdot d\ell_2}{\mathscr{r}} \quad , \quad L = \frac{1}{c^2}\oint \frac{d\ell' \cdot d\ell}{\mathscr{r}} \quad ,$$

which turn out to be impractical in most cases for computing inductances. Far easier is the procedure resembling the calculation of capacitance: run a current, calculate the ***B*** produced, then the flux, and then use

$$\Phi_{B1} = cMI_2 \quad \text{or} \quad \Phi_B = cLI \quad ,$$

as we did in the example of the coupled circular loops.

Calculation of inductance (continued)

For example: what's the inductance per unit length of a very long solenoid with length ℓ, radius a, and turns per unit length n? First run a current I through it, then the magnetic field inside is

$$B = \frac{4\pi}{c} nI$$

$$\Rightarrow \quad \Phi_B = BA = \frac{4\pi}{c} nI \cdot n\ell\pi a^2 = \frac{4\pi^2 a^2 n^2 \ell}{c} I = cLI$$

$$\Rightarrow \quad L = \frac{4\pi^2 a^2 n^2 \ell}{c^2} \quad \left[= \mu_0 \pi a^2 n^2 \ell \text{ in MKS} \right].$$