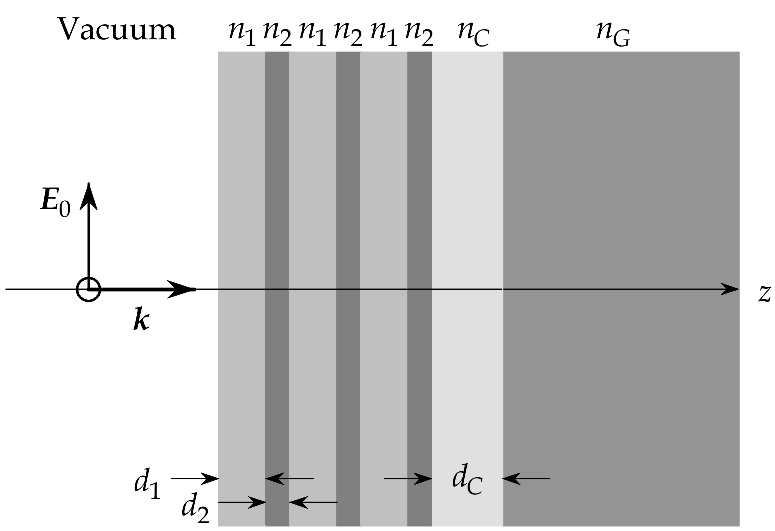
Physics 218 Makeup Midterm Exam: Solutions

March 2004

If any of these solutions seems obscure, please contact us so we can explain it better.

Problem 1 (30 points)

Light with wavelength λ_0 is incident from vacuum normally on a plane-parallel multilayer dielectric structure, which covers a semi-infinite region made of glass with refractive index n_G. The first six layers are three identical pairs of layers of material with indices n_1 and *n_2, and thicknesses $d_1 = \lambda_0/2n_1$* and *$d_2 = \lambda_0/2n_2$. Between these and the glass is a seventh layer, with index $n_C = \sqrt{n_G}$ and thickness $d_C = \lambda_0/4n_C$.*



Calculate the fraction of the incident intensity transmitted into the glass, $\tau = I_T/I_I$.

Consider the first six layers first. The phase delays across them are all the same:

$$\delta_1 = kd = \frac{2\pi n_1}{\lambda_0}\frac{\lambda_0}{2n_1} = \pi = \delta_2 \quad ,$$

$$\cos\delta_1 = -1 = \cos\delta_2 \quad ,$$
$$\sin\delta_1 = 0 = \sin\delta_2 \quad ,$$

and since the sine terms are all zero we don't even have to note the admittances to write down the characteristic matrices:

$$M_1 = M_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad .$$

Now the seventh layer. Its admittance is:

$$Y_C = \sqrt{\varepsilon_C} = n_C \quad .$$

The phase delay for propagation across is

$$\delta_C = kd = \frac{2\pi n_C}{\lambda_0}\frac{\lambda_0}{4n_C} = \frac{\pi}{2} \quad ,$$

so

$$\cos\delta_1 = 0 = \cos\delta_2 \quad ,$$
$$\sin\delta_1 = 1 = \sin\delta_2 \quad .$$

Its characteristic matrix is therefore

$$M_C = \begin{pmatrix} 0 & -\dfrac{i}{n_C} \\ -in_C & 0 \end{pmatrix} \quad .$$

Thus the characteristic matrix of the layer stack is

$$M = (M_1 M_2)^3 M_C = M_C = \begin{pmatrix} 0 & -\dfrac{i}{n_C} \\ -in_C & 0 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \quad .$$

The admittances of the incident and final media are $Y_0 = 1$ (vacuum) and $Y_G = n_G$ (glass), so the amplitude transmission coefficient is

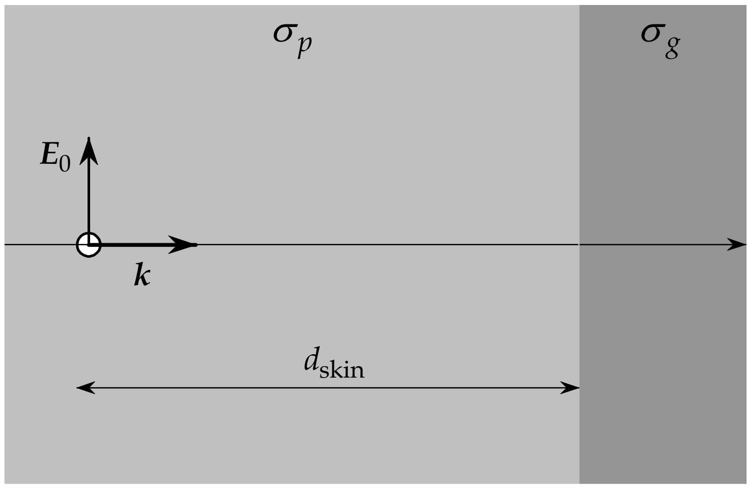
$$t = \frac{2Y_0}{m_{11}Y_0 + m_{12}Y_0 Y_G + m_{21} + m_{22}Y_G} = \frac{2}{-\dfrac{in_G}{n_C} - in_C} = -\frac{1}{in_C} \quad ,$$

and the intensity transmission coefficient is

$$\tau = \frac{Y_3}{Y_0} tt^* = n_G \frac{1}{-in_C}\frac{1}{in_C} \boxed{=1} \; .$$

Problem 2 (40 points)

You are located within a poor conductor, with conductivity σ_p*, and shine collimated (i.e. plane-wave) light with frequency* ω *at normal incidence at the surface of a good conductor, with conductivity* σ_g*. The surface of the good conductor lies one skin depth away from you.*



Some of the light you shine reflects back to you. Calculate the ratio of intensities of the light that makes it back to you, and the light that left your laser.

By definition of skin depth: by the time the wave gets to the surface of the good conductor, the fields have decreased in amplitude by a factor of $1/e$ from their original value (E_0), and by the time it gets back they have decreased by another factor of $1/e$ from the reflected value, to a value we shall call E_1. It remains to find the reflection coefficicent of the conducting surface. Continuity of the parallel components of $\boldsymbol{E}$ and $\boldsymbol{H}$ gives us

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \quad ,$$

$$\frac{1}{\mu_p}\left(\frac{c\tilde{k}_p}{\omega}\tilde{E}_{0I} - \frac{c\tilde{k}_p}{\omega}\tilde{E}_{0R}\right) = \frac{1}{\mu_g}\frac{c\tilde{k}_g}{\omega}\tilde{E}_{0T} \quad ,$$

where the $\tilde{k}$s are the complex wavevectors. Define

$$\tilde{\beta} = \frac{\mu_p \tilde{k}_g}{\mu_g \tilde{k}_p} = \frac{1}{\sqrt{\mu_p \varepsilon_p}\,\dfrac{\omega}{c} + i\,\dfrac{2\pi\sigma_p}{c}\sqrt{\dfrac{\mu_p}{\varepsilon_p}}}\,\frac{\sqrt{2\pi\omega\mu_g\sigma_g}}{c\mu_g}(1+i)$$

$$\cong \frac{c}{\omega\sqrt{\mu_p\varepsilon_p}}\left[1 - i\frac{2\pi\sigma_p}{\varepsilon_p\omega}\right]\left[\sqrt{\frac{2\pi\sigma_g}{\mu_p\mu_g\varepsilon_p\omega}}(1+i)\right] \quad .$$

If this were multiplied out, all the products involving the imaginary part of the first [ ] would appear added with products involving the imaginary part of the second [ ] – with respect to which they are negligible. Thus

$$\tilde{\beta} \cong \sqrt{\frac{2\pi\sigma_g}{\mu_p\mu_g\varepsilon_p\omega}}(1+i) \equiv \gamma(1+i) \quad ,$$

so

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \quad ,$$
$$\tilde{E}_{0I} - \tilde{E}_{0R} = \tilde{\beta}\tilde{E}_{0T} \quad ,$$

from which the amplitude reflection coefficient comes out as

$$\tilde{E}_{0R} = \frac{1-\tilde{\beta}}{1+\tilde{\beta}}\tilde{E}_{0I} \quad ,$$

as usual. So,

$$\frac{\tilde{E}_1}{\tilde{E}_0} = \frac{1}{e^2}\frac{1-\tilde{\beta}}{1+\tilde{\beta}} \quad \text{, and}$$

$$\frac{I_1}{I_0} = \frac{1}{e^4}\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\frac{1-\tilde{\beta}^*}{1+\tilde{\beta}^*} = \frac{1}{e^4}\frac{1-\left(\tilde{\beta}+\tilde{\beta}^*\right)+\tilde{\beta}\tilde{\beta}^*}{1-\left(\tilde{\beta}+\tilde{\beta}^*\right)+\tilde{\beta}\tilde{\beta}^*}$$

$$= \boxed{\frac{1}{e^4}\frac{1-2\gamma+2\gamma^2}{1-2\gamma+2\gamma^2} = \frac{1}{e^4}\frac{1-2\sqrt{\dfrac{2\pi\sigma_g}{\mu_p\mu_g\varepsilon_p\omega}}+\dfrac{4\pi\sigma_g}{\mu_p\mu_g\varepsilon_p\omega}}{1+2\sqrt{\dfrac{2\pi\sigma_g}{\mu_p\mu_g\varepsilon_p\omega}}+\dfrac{4\pi\sigma_g}{\mu_p\mu_g\varepsilon_p\omega}}} \;.$$

Change all the factors of 4π to 1, and apply the appropriate definitions of the permeability and permittivity (e.g. $\mu_1 = \mu_{1r}\mu_0$), to get the MKS version of the answer.

Yes, this is the same as the reflectivity of a good conductor in vacuum, apart from the factors of $1/e$.

Problem 3 (30 points)

Consider the low-frequency part of the normal-dispersion domain for a dispersive medium; that is, for

$$n = 1 + \frac{2\pi N q^2}{m_e} \sum_{j=1}^{M} \frac{f_j\left(\omega_j^2 - \omega^2\right)}{\left(\omega_j^2 - \omega^2\right)^2 + \gamma_j^2 \omega^2} \quad ,$$

(cgs units; change 2π to $1/2\varepsilon_0$ *for MKS) we are concerned with frequencies that lie far enough from the resonances to ignore damping.*

a. *Show that under these conditions the expression for the index can be simplified to good approximation as*

$$n = 1 + \frac{2\pi N q^2}{m_e} \sum_{j=1}^{M} \frac{f_j}{\omega_j^2 - \omega^2} \quad .$$

First,

$$n = 1 + \frac{2\pi N q^2}{m_e} \sum_{j=1}^{M} \frac{f_j}{\left(\omega_j^2 - \omega^2\right) + \dfrac{\gamma_j^2 \omega^2}{\left(\omega_j^2 - \omega^2\right)}} \quad .$$

If ω is far enough from each ω_j that damping is negligible, then by definition $\gamma_j \omega \ll \left|\omega_j - \omega\right|$, so we can neglect the second term in the denominator under the sum:

$$n = 1 + \frac{2\pi N q^2}{m_e} \sum_{j=1}^{M} \frac{f_j}{\omega_j^2 - \omega^2} \quad \text{,q.e.d.}$$

b. *Show that for frequencies* $\omega < \omega_j$ *for all j, the index takes the form*

$$n = 1 + A\left(1 + \frac{B}{\lambda^2}\right) \quad .$$

Find expressions for A and B. (This result is called Cauchy's formula.*)*

Note that

$$\frac{1}{\omega_j^2-\omega^2}=\frac{1}{\omega_j^2}\left(1-\frac{\omega^2}{\omega_j^2}\right)^{-1} \quad .$$

If $\omega<\omega_j$ for all *j*, by enough that damping in negligible (i.e. by many widths of the absorptions), then it must generally be true that $\omega^2 \ll \omega_j^2$, so we can use $(1+x)^a \cong 1+ax$:

$$\frac{1}{\omega_j^2-\omega^2}\cong\frac{1}{\omega_j^2}\left(1+\frac{\omega^2}{\omega_j^2}\right) \quad ,$$

whence

$$n=1+\frac{2\pi Nq^2}{m_e}\sum_{j=1}^{M}\frac{f_j}{\omega_j^2}+\frac{2\pi Nq^2}{m_e}\omega^2\sum_{j=1}^{M}\frac{f_j}{\omega_j^4} \quad .$$

Now use $\omega=2\pi c/\lambda$, and abbreviate

$$A=\frac{2\pi Nq^2}{m_e}\sum_{j=1}^{M}\frac{f_j}{\omega_j^2} \quad ,$$

$$B=(2\pi)^2 c^2\,\frac{\sum_{j=1}^{M} f_j/\omega_j^4}{\sum_{j=1}^{M} f_j/\omega_j^2} \quad ,$$

and we get Cauchy's formula,

$$n=1+A\left(1+\frac{B}{\lambda^2}\right) \quad \text{, q.e.d.}$$
