Today in Physics 218: updates for other tools in electrodynamics

- From last time: Symmetry of the equations: magnetic monopoles?
- The Maxwell equations in matter
- Boundary conditions for electrodynamics
- Potentials in electrodynamics

Note: Monday's class is cancelled, in honor of this gentleman on what would have been his 75th birthday.

The Maxwell equations

Again, here are the Maxwell equations, in vacuum, in final form:

$$\nabla\cdot\boldsymbol{E} = 4\pi\rho \qquad \nabla\cdot\boldsymbol{B} = 0$$
$$\nabla\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t} \qquad \nabla\times\boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}$$

in cgs units, or

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla\cdot\boldsymbol{B} = 0$$
$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \qquad \nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}$$

in MKS units.

Magnetic monopoles

The only remaining sense in which these equations may still be approximate is if magnetic charges (monopoles) exist. We will see a powerful argument for searching for magnetic monopoles in the first homework set (Griffiths problem 8.12); they would also symmetrize the Maxwell equations. Note that if there are no electric charges or currents, the Maxwell equations are symmetrical:

$$\nabla\cdot\boldsymbol{E} = 0 \qquad \nabla\cdot\boldsymbol{B} = 0$$
$$\nabla\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t} \qquad \nabla\times\boldsymbol{B} = \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}$$

Magnetic monopoles (continued)

If, on the other hand, there *were* magnetic as well as electric monopoles, with magnetic charge density η and magnetic current density $\boldsymbol{K}$, then we'd have

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho \qquad \nabla \cdot \boldsymbol{B} = 4\pi\eta$$
$$\nabla \times \boldsymbol{E} = -\frac{4\pi}{c}\boldsymbol{K} - \frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}$$

where, if both electric and magnetic charge were conserved,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$
$$\frac{\partial \eta}{\partial t} + \nabla \cdot \boldsymbol{K} = 0$$

Update #1: the Maxwell equations in matter

Those who took PHY 217 last semester didn't discuss polarization and magnetization of matter, and thus won't be familiar with the following. Don't worry; we will only be using linear media this semester, and the general forms are presented here only for reference, and for the edification of those who took PHY217 last year.

Charge density comes in free or bound form, bound charges being related to polarization, $\boldsymbol{P}$:

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \boldsymbol{P}$$

Current density comes in free and bound form (the latter related to the magnetization $\boldsymbol{M}$), plus one other that arises from our new consideration of time-variable charge density.

Maxwell equations in matter (continued)

A time-varying free charge density leads to a time-varying free current density, through the conservation of charge. A time-varying *bound* charge density similarly leads to a current density that has nothing to do either with free or bound currents:

$$\nabla \cdot \boldsymbol{J}_p = -\frac{\partial}{\partial t}\rho_b = \frac{\partial}{\partial t}\nabla \cdot \boldsymbol{P} = \nabla \cdot \frac{\partial \boldsymbol{P}}{\partial t} \quad ,$$

or $\boldsymbol{J}_p = \dfrac{\partial \boldsymbol{P}}{\partial t}$.

Thus, $\boldsymbol{J} = \boldsymbol{J}_f + \boldsymbol{J}_b + \boldsymbol{J}_p = \boldsymbol{J}_f + c\nabla \times \boldsymbol{M} + \dfrac{\partial \boldsymbol{P}}{\partial t}$.

Maxwell equations in matter (continued)

In PHY 217 we defined the auxiliary fields **D** and **H** as:

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} \quad , \qquad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} \quad .$$

So let's put the expressions for charge and current density into the complete Maxwell equations and rearrange using the auxiliary fields:

$$\nabla\cdot\mathbf{E} = 4\pi\rho_f - 4\pi\nabla\cdot\mathbf{P} \qquad \nabla\cdot\mathbf{B} = 0$$

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \qquad \nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J}_f + 4\pi\nabla\times\mathbf{M} + \frac{4\pi}{c}\frac{\partial\mathbf{P}}{\partial t} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}$$

Maxwell equations in matter (continued)

or

$$\boxed{\begin{array}{ll}\nabla\cdot\mathbf{D} = 4\pi\rho_f & \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} & \nabla\times\mathbf{H} = \frac{4\pi}{c}\mathbf{J}_f + \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}\end{array}}$$

in cgs units;

$$\nabla\cdot\mathbf{D} = \rho_f \qquad \nabla\cdot\mathbf{B} = 0$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad \nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t}$$

in MKS. (Again, don't worry; we won't be using **D** and **H** to do problems this semester.)

Update #2: boundary conditions

In PHY 217, whenever we learned a new Maxwell equation, we used it to determine boundary conditions: that is, the influence of charge or current densities on the fields and their derivatives, for use in boundary-value problems. It's easier for this to work with the integral form of the equations:

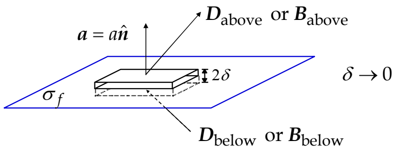
$$\oint\mathbf{D}\cdot d\mathbf{a} = 4\pi Q_{f,encl} \qquad \oint\mathbf{B}\cdot d\mathbf{a} = 0$$

$$\oint\mathbf{E}\cdot d\boldsymbol{\ell} = -\frac{1}{c}\frac{d}{dt}\int\mathbf{B}\cdot d\mathbf{a} \qquad \oint\mathbf{H}\cdot d\boldsymbol{\ell} = \frac{4\pi}{c}I_{f,encl} + \frac{1}{c}\frac{d}{dt}\int\mathbf{D}\cdot d\mathbf{a}$$

*Again, do not fear the appearance here of **D** and **H**; you may translate them for purposes this semester as $\varepsilon\mathbf{E}$ and $\mathbf{B}/\mu$, or, if you took the class last year, note that we're deriving the boundary conditions completely generally.*

Boundary conditions

Consider the application of these relations to a boundary surface with free charge density σ_f and free surface current density $\boldsymbol{K}_f$, over a scale small enough that the surface looks flat, but large enough that charge quantization is averaged out. First, construct a Gaussian surface with flat faces (area $\boldsymbol{a}$) parallel to the surface, infinitesimally above and below the surface:



Boundary conditions (continued)

Then, since the flux through the sides is negligible,

$$\oint \boldsymbol{D}\cdot d\boldsymbol{a} = 4\pi Q_{f,encl}$$

$$\left(D_{\perp,\text{ above}} - D_{\perp,\text{ below}}\right)a = 4\pi\sigma_f a$$

$$\Rightarrow \quad \left(D_{\perp,\text{ above}} - D_{\perp,\text{ below}}\right) = 4\pi\sigma_f$$

The charge sheet makes a discontinuity of $4\pi\sigma_f$ in $D_\perp$. Similarly, since there is no such thing (yet) as magnetic charge,

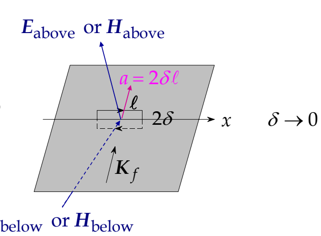
$$\left(B_{\perp,\text{ above}} - B_{\perp,\text{ below}}\right) = 0 \quad .$$

So far this is the same as in quasistatics.

Boundary conditions (continued)

Next consider a rectangular Ampèrean loop enclosing some of the surface current: (infinitesimal) height 2δ, width ℓ, long sides parallel to the surface, and area vector $\boldsymbol{a}$ parallel to the surface:

Define a vector $\boldsymbol{\ell}$ equal in length to the loop width, pointing in the $+x$ direction.



Boundary conditions (continued)

First apply Ampère's law. The flux of $\boldsymbol{D}$ through the loop approaches zero as $\delta \to 0$, so

$$\oint \boldsymbol{H} \cdot d\boldsymbol{\ell} = \frac{4\pi}{c} I_{f,encl} + \frac{1}{c}\frac{d}{dt}\int \boldsymbol{D} \cdot d\boldsymbol{a}$$
$$= \frac{4\pi}{c} \ell \boldsymbol{K}_f \cdot \hat{\boldsymbol{a}} = \frac{4\pi}{c} \boldsymbol{K}_f \cdot \left(\hat{\boldsymbol{n}} \times \boldsymbol{\ell}\right) \quad ,$$

where $\hat{\boldsymbol{n}}$ is the unit vector normal to the surface, as before. In the line integral we can ignore the sides as $\delta \to 0$, so

$$\boldsymbol{H}_{\text{above}} \cdot \boldsymbol{\ell} - \boldsymbol{H}_{\text{below}} \cdot \boldsymbol{\ell} = \frac{4\pi}{c} \boldsymbol{K}_f \cdot \left(\hat{\boldsymbol{n}} \times \boldsymbol{\ell}\right) = \frac{4\pi}{c} \boldsymbol{\ell} \cdot \left(\boldsymbol{K}_f \times \hat{\boldsymbol{n}}\right)$$

Triple-product rule #1

$$\Rightarrow \quad \boldsymbol{H}_{\text{above}} - \boldsymbol{H}_{\text{below}} = \frac{4\pi}{c} \boldsymbol{K}_f \times \hat{\boldsymbol{n}} \quad .$$

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Boundary conditions (continued)

We can use the same loop and current, and apply Faraday's law, and since the magnetic flux vanishes as δ does,

$$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{1}{c}\frac{d}{dt}\int \boldsymbol{B} \cdot d\boldsymbol{a}$$
$$\boldsymbol{E}_{\text{above}} \cdot \boldsymbol{\ell} - \boldsymbol{E}_{\text{below}} \cdot \boldsymbol{\ell} = 0$$

Thus, $E_{\|,\text{above}} - E_{\|,\text{below}} = 0$.

Summary: when traversing a surface with free charges and currents,

$B_\perp$ and $E_\|$ are continuous;
$D_\perp$ is discontinuous by $4\pi\sigma_f$;
$H_\|$ is discontinuous by $\left(4\pi/c\right)\boldsymbol{K}_f \times \hat{\boldsymbol{n}}$.

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Boundary conditions (continued)

In linear media, $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ and $\boldsymbol{H} = \boldsymbol{B}/\mu$, and if we insert these into the boundary conditions we just obtained, we get a set of boundary conditions we can all use:

$$\boxed{\begin{aligned} &\varepsilon_{\text{above}} E_{\perp,\text{above}} - \varepsilon_{\text{below}} E_{\perp,\text{below}} = 4\pi\sigma_f \\ &B_{\perp,\text{above}} - B_{\perp,\text{below}} = 0 \\ &E_{\|,\text{above}} - E_{\|,\text{below}} = 0 \\ &\frac{1}{\mu_{\text{above}}} B_{\|,\text{above}} - \frac{1}{\mu_{\text{below}}} B_{\|,\text{below}} = \frac{4\pi}{c}\left|\boldsymbol{K}_f \times \hat{\boldsymbol{n}}\right| \end{aligned}}$$

and we will, in fact, use them in a few weeks, when we discuss the reflection and refraction of light by material surfaces.

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Update #3: potentials

In electrodynamics the divergence of $\boldsymbol{B}$ is still zero, so according to the Helmholtz theorem and its corollaries (#2, in this case), we can still define a magnetic vector potential as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad .$$

However, the curl of $\boldsymbol{E}$ isn't zero; in fact it hasn't been since we started magnetoquasistatics. What does this imply for the electric potential? Note that Faraday's law can be put in a suggestive form:

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \boldsymbol{A}) \quad \text{, or}$$

$$\nabla \times \left(\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \right) = 0 \quad .$$

Potentials (continued)

Thus Corollary #1 to the Helmholtz theorem allows us to define a scalar potential for that last bracketed term:

$$\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = -\nabla V \quad \Rightarrow \quad \boldsymbol{E} = -\nabla V - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$$

so we can still use the scalar electric potential in electrodynamics, but now both the scalar and the vector potential must be used to determine $\boldsymbol{E}$.

"Reference points" for potentials

Our usual reference point for the scalar potential in electrostatics is $V \to 0$ at $r \to \infty$. For the vector potential in magnetostatics we imposed the condition $\nabla \cdot \boldsymbol{A} = 0$.

- These reference points arise from exploitation of the built-in ambiguities in the static potentials: one can add any gradient to $\boldsymbol{A}$ and any constant to V, and still get the same fields.
- So we decided to add whatever was necessary to make the second-order differential equations in $\boldsymbol{A}$ and V look like Poisson's equation (i.e. easy to solve).

In electrodynamics these choices no longer produce that last result:

"Reference points" for potentials (continued)

For instance, Gauss's law gives us

$$\nabla\cdot\boldsymbol{E} = 4\pi\rho \quad \Rightarrow \quad \nabla\cdot\left(-\nabla V - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}\right) = 4\pi\rho$$

$$\Rightarrow \nabla^2 V + \frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot\boldsymbol{A} = -4\pi\rho \quad ,$$

which with $\nabla\cdot\boldsymbol{A} = 0$ still leaves us with a Poisson equation, but Ampère's law gives

$$\nabla\times(\nabla\times\boldsymbol{A}) = \frac{4\pi}{c}\boldsymbol{J} - \frac{1}{c}\frac{\partial}{\partial t}\nabla V - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2}$$

$$\nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A} = \qquad \text{(P.R. \#11)}$$

or $$\left(\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2}\right) - \nabla\left(\nabla\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial V}{\partial t}\right) = -\frac{4\pi}{c}\boldsymbol{J} \quad .$$

"Reference points" for potentials (continued)

This latter equation does not of course reduce to a Poisson equation with any of the reference conditions we have imposed. Thus we must look harder to use the built-in ambiguity of the potentials to make the differential equations simpler. The general way to do this, which we will cover next time, is called a gauge transformation.