Today in Physics 218: gauge transformations

More updates as we move from quasistatics to dynamics:

- #3: potentials
- For better use of potentials: gauge transformations
- The Coulomb and Lorentz gauges
- #4: force, energy, and momentum in electrodynamics

The spectre of the Brocken. Photo by Galen Rowell.

Update #3: potentials

In electrodynamics the divergence of $\boldsymbol{B}$ is still zero, so according to the Helmholtz theorem and its corollaries (#2, in this case), we can still define a magnetic vector potential as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad .$$

However, the curl of $\boldsymbol{E}$ isn't zero; in fact it hasn't been since we started magnetoquasistatics. What does this imply for the electric potential? Note that Faraday's law can be put in a suggestive form:

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\nabla \times \boldsymbol{A}\right) \quad \text{, or}$$

$$\nabla \times \left(\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \right) = 0 \quad .$$

Potentials (continued)

Thus Corollary #1 to the Helmholtz theorem allows us to define a scalar potential for that last bracketed term:

$$\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = -\nabla V \quad \Rightarrow \quad \boldsymbol{E} = -\nabla V - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$$

so we can still use the scalar electric potential in electrodynamics, but now both the scalar and the vector potential must be used to determine $\boldsymbol{E}$.

"Reference points" for potentials

Our usual reference point for the scalar potential in electrostatics is $V \to 0$ at $r \to \infty$. For the vector potential in magnetostatics we imposed the condition $\nabla \cdot \boldsymbol{A} = 0$.

- These reference points arise from exploitation of the built-in ambiguities in the static potentials: one can add any gradient to $\boldsymbol{A}$ and any constant to V, and still get the same fields.
- So we decided to add whatever was necessary to make the second-order differential equations in $\boldsymbol{A}$ and V look like Poisson's equation (i.e. easy to solve).

In electrodynamics these choices no longer produce that last result:

“Reference points” for potentials (continued)

For instance, Gauss’s law gives us

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho \quad \Rightarrow \quad \nabla \cdot \left(-\nabla V - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \right) = 4\pi\rho$$

$$\Rightarrow \nabla^2 V + \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{A} = -4\pi\rho \quad ,$$

which with $\nabla \cdot \boldsymbol{A} = 0$ still leaves us with a Poisson equation, but Ampère’s law gives

$$\nabla \times \left(\nabla \times \boldsymbol{A} \right) = \frac{4\pi}{c}\boldsymbol{J} - \frac{1}{c}\frac{\partial}{\partial t}\nabla V - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2}$$

$$\nabla \left(\nabla \cdot \boldsymbol{A} \right) - \nabla^2 \boldsymbol{A} = \qquad \text{(P.R. \#11)}$$

or $$\left(\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} \right) - \nabla \left(\nabla \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial V}{\partial t} \right) = -\frac{4\pi}{c}\boldsymbol{J} \quad .$$

“Reference points” for potentials (continued)

This latter equation does not of course reduce to a Poisson equation with any of the reference conditions we have imposed. Thus we must look harder to use the built-in ambiguity of the potentials to make the differential equations simpler. The general way to do this, which we will cover next time, is called a gauge transformation.

Gauge transformations

In electro- and magentostatics, we showed that we could always choose our conventional reference points,

$$V \to 0 \text{ as } r \to \infty \qquad\qquad \nabla \cdot \boldsymbol{A} = 0$$

without placing any peculiar constraints on $\boldsymbol{E}$ or $\boldsymbol{B}$. Now we have two, more complicated equations to simplify, and a more general approach is more fruitful.

Consider performing a transformation on $\boldsymbol{A}$ and V: add a vector to $\boldsymbol{A}$ and a scalar to V, giving *new* potential functions:

$$\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\alpha} \qquad V' = V + \beta$$

Gauge transformations (continued)

Now, we can't add just any old thing to the potentials; we need for the *fields* arising from the new potentials to be the same as those from the old:

$$\mathbf{B}' = \mathbf{B}$$

$$\nabla \times \mathbf{A}' = \nabla \times \mathbf{A} + \nabla \times \boldsymbol{\alpha}$$

$$\Downarrow$$

$$\nabla \times \boldsymbol{\alpha} = 0 \quad , \quad \text{or}$$

$$\boldsymbol{\alpha} = \nabla \lambda' \quad ,$$

$$\mathbf{E}' = \mathbf{E}$$

$$\nabla V' = \nabla V + \nabla \beta$$

$$-\mathbf{E}' - \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t} = -\mathbf{E} - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \nabla \beta$$

$$\nabla \beta = \frac{1}{c}\frac{\partial}{\partial t}\left(\mathbf{A} - \mathbf{A}'\right) = -\frac{1}{c}\frac{\partial \boldsymbol{\alpha}}{\partial t}$$

where λ' is a scalar function of $\boldsymbol{r}$ and t.

Gauge transformations (continued)

Combine those last two results:

$$\nabla\beta = -\frac{1}{c}\frac{\partial}{\partial t}\nabla\lambda'$$

$$\nabla\left(\beta + \frac{1}{c}\frac{\partial\lambda'}{\partial t}\right) = 0$$

and integrate the second one over volume, applying the fundamental theorem of calculus:

$$\beta + \frac{1}{c}\frac{\partial\lambda'}{\partial t} = f(t)$$

The integration "constant" f is not a function of position

Gauge transformations (continued)

We can combine the integration "constant" f with λ' by defining

$$\lambda = \lambda' - c\int_0^t f(t')\,dt$$

Then, $\beta = -\frac{1}{c}\frac{\partial\lambda'}{\partial t} + f(t) = -\frac{1}{c}\left(\frac{\partial\lambda}{\partial t} + cf(t)\right) + f(t) = -\frac{1}{c}\frac{\partial\lambda}{\partial t}$

$\alpha = \nabla\lambda' = \nabla\lambda \quad ,$

Thus for any scalar function $\lambda = \lambda(\boldsymbol{r}, t)$, the transformation

$$\boxed{V' = V - \frac{1}{c}\frac{\partial\lambda}{\partial t} \qquad \boldsymbol{A}' = \boldsymbol{A} + \nabla\lambda}$$

Gauge transformation

makes new potentials but leaves the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ unchanged.

Gauge transformations (continued)

This sort of operation on potentials is called a gauge transformation, and a particular choice of λ is called a gauge condition.

- Clever choices of λ can simplify one or the other of the second-order differential equations for the potentials.
- The solution of these simpler equations for the transformed potentials gives the same fields as the solutions to the untransformed, complicated equations

For instance, to simplify $\nabla^2 V + \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{A} = -4\pi\rho$,
we could pick λ such that

$$\boxed{\nabla \cdot \boldsymbol{A} = 0} \qquad \text{Coulomb gauge}$$

Gauge transformations (continued)

which, as we showed in PHY 217 (see

http://www.pas.rochester.edu/~dmw/phy217/Lectures/Lect_28b.pdf)

we can always do; that is, there always exists a function λ that we can add to $\boldsymbol{A}$ to give an $\boldsymbol{A}'$ with zero divergence. With this gauge condition,

$$\nabla^2 V' = -4\pi\rho \quad ,$$

just like in electrostatics (hence the name).

- Coulomb gauge only does a lot of good in magnetoquasistatics, because otherwise the time derivative of $\boldsymbol{A}$ gets big enough that you have to remember that

$$\boldsymbol{E} = -\nabla V - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \quad .$$

Lorentz gauge

It's hard to compute $\boldsymbol{A}$ in Coulomb gauge. On the other hand, we could choose λ such that

$$\boxed{\nabla \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial V}{\partial t} = 0 \quad ,}$$

Lorentz gauge

for which the second-order PDEs we saw several pages back become

$$\nabla^2 V + \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{A} = -4\pi\rho$$

$$\Rightarrow \quad \boxed{\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -4\pi\rho}$$

Lorentz gauge (continued)

and
$$\left(\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial V}{\partial t}\right) = -\frac{4\pi}{c}\boldsymbol{J}$$
$$\Rightarrow \boxed{\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\frac{4\pi}{c}\boldsymbol{J}}$$

- Not utterly simple, but at least V and $\boldsymbol{A}$ are separately determined, and the four equations are very similar to one another.
- In fact, the four second-order PDEs here are (inhomogeneous) **wave equations**, the solution of which will concern us for the bulk of this semester.

Lorentz gauge (continued)

Can one always use the Lorentz gauge? I think so, because:

$$\nabla \cdot \boldsymbol{A}' + \frac{1}{c}\frac{\partial V'}{\partial t} = 0$$

$$\nabla \cdot \left(\boldsymbol{A} + \nabla \lambda \right) + \frac{1}{c}\frac{\partial}{\partial t}\left(V - \frac{1}{c}\frac{\partial \lambda}{\partial t} \right) = 0$$

$$\nabla^2 \lambda - \frac{1}{c^2}\frac{\partial^2 \lambda}{\partial t^2} = -\left(\nabla \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial V}{\partial t} \right) = g\left(\boldsymbol{r}, t \right)$$

That is, the Lorentz gauge condition λ always obeys an inhomogenous wave equation, just as do the potentials *in* Lorentz gauge. In MTH 281 you proved the existence of solutions to such equations; we'll demonstrate such solutions this semester.

Update #4: force, energy, and momentum in electrodynamics

The Lorentz force law,

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right)$$

hasn't changed since we first learned it, but can be used to illuminate the relation of the potentials to mechanics. To wit:

$$\boldsymbol{F} = q\left(-\nabla V - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} + \frac{1}{c}\boldsymbol{v}\times\nabla\times\boldsymbol{A}\right)$$

According to product rule #4, written for $\boldsymbol{v}$ and $\boldsymbol{A}$,

$$\begin{aligned}\nabla\left(\boldsymbol{v}\cdot\boldsymbol{A}\right) &= \boldsymbol{v}\times\left(\nabla\times\boldsymbol{A}\right) + \cancelto{0}{\boldsymbol{A}\times\left(\nabla\times\boldsymbol{v}\right)} + \left(\boldsymbol{v}\cdot\nabla\right)\boldsymbol{A} + \cancelto{0}{\left(\boldsymbol{A}\cdot\nabla\right)\boldsymbol{v}}\\ &= \boldsymbol{v}\times\left(\nabla\times\boldsymbol{A}\right) + \left(\boldsymbol{v}\cdot\nabla\right)\boldsymbol{A}\end{aligned}$$

because $\boldsymbol{v}$ doesn't depend explicitly on $\boldsymbol{r}$

Force, energy, and momentum in electrodynamics (continued)

so

$$\boldsymbol{F} = -q\left(\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} + \frac{1}{c}(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{A} + \nabla\left(V - \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A}\right)\right)$$

It will be useful to introduce total time derivatives:

$$\frac{d\boldsymbol{A}}{dt} = \frac{\partial \boldsymbol{A}}{\partial t} + \frac{\partial \boldsymbol{A}}{\partial x}\frac{dx}{dt} + \frac{\partial \boldsymbol{A}}{\partial y}\frac{dy}{dt} + \frac{\partial \boldsymbol{A}}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial \boldsymbol{A}}{\partial t} + \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)\boldsymbol{A} = \frac{\partial \boldsymbol{A}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{A}$$

so

$$\boldsymbol{F} = -q\left(\frac{1}{c}\frac{d\boldsymbol{A}}{dt} + \nabla\left(V - \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A}\right)\right)$$

Force, energy, and momentum in electrodynamics (continued)

But $\boldsymbol{F} = d\boldsymbol{p}/dt$, where $\boldsymbol{p}$ is the momentum of the point charge q, so

$$\frac{d\boldsymbol{p}}{dt} = -q\left(\frac{1}{c}\frac{d\boldsymbol{A}}{dt} + \nabla\left(V - \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A}\right)\right)$$

$$\frac{d}{dt}\left(\boldsymbol{p} + \frac{q}{c}\boldsymbol{A}\right) = -\nabla\left(q\left[V - \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A}\right]\right)$$

This has the form $\boldsymbol{F} = \frac{d}{dt}\boldsymbol{p}_{\text{canonical}} = -\nabla U$, where the canonical momentum is

$$\boldsymbol{p}_{\text{canonical}} = \boldsymbol{p} + \frac{q}{c}\boldsymbol{A} \quad ,$$

and the related potential energy is

$$U = q\left(V - \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A}\right) \quad .$$

Since $\boldsymbol{v} = d\boldsymbol{r}/dt$, $\boldsymbol{r}$ is the conjugate canonical coordinate

Force, energy, and momentum in electrodynamics (continued)

In quantum mechanics one normally uses the Hamiltonian formulation of dynamics; this last result represents the easiest way to incorporate electrodynamics into quantum mechanics. From correspondence to classical mechanics, the Hamiltonian

$$H = \dot{\boldsymbol{x}}_{\text{canonical}} \cdot \boldsymbol{p}_{\text{canonical}} - L$$

gives rise to the quantum-mechanical equations of motion,

$$H\psi = E\psi \quad ,$$

as well as the classical ones,

$$\frac{\partial H}{\partial p_{\text{canonical}}} = \dot{x}_{\text{canonical}} \quad , \quad \frac{\partial H}{\partial x_{\text{canonical}}} = \dot{p}_{\text{canonical}} \quad .$$

Force, energy, and momentum in electrodynamics (continued)

The classical Lagrangian for a charge q in an electromagnetic field is therefore

$$L = \frac{1}{2} m \dot{x}_{\text{canonical}}{}^2 - U = \frac{1}{2} m v^2 - qV + \frac{q}{c} \boldsymbol{v} \cdot \boldsymbol{A} \quad ,$$

so the classical Hamiltonian is

$$H = \dot{\boldsymbol{x}}_{\text{canonical}} \cdot \boldsymbol{p}_{\text{canonical}} - L = m v^2 + \frac{q}{c} \boldsymbol{v} \cdot \boldsymbol{A} - \frac{1}{2} m v^2 + qV - \frac{q}{c} \boldsymbol{v} \cdot \boldsymbol{A}$$

$$= \frac{1}{2} m v^2 + qV = \frac{p^2}{2m} + qV = \boxed{\frac{1}{2m} \left| \boldsymbol{p}_{\text{canonical}} - \frac{q}{c} \boldsymbol{A} \right|^2 + qV} \quad .$$

In quantum mechanics, $\boldsymbol{p}_{\text{canonical}} \rightarrow -i\hbar\nabla$.