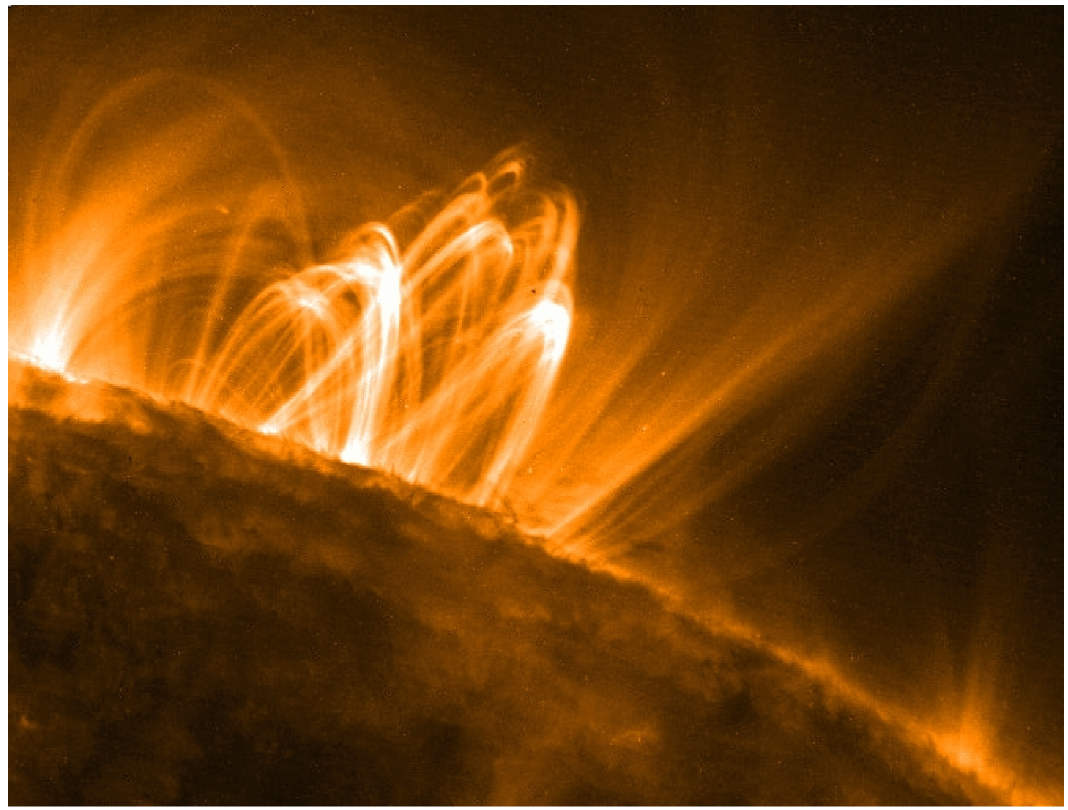
Today in Physics 218: the classic conservation laws in electrodynamics

- ❑ Poynting's theorem
- ❑ Energy conservation in electrodynamics
- ❑ The Maxwell stress tensor (which gets rather messy)
- ❑ Momentum conservation in electrodynamics



Electromagnetism on the sun, doing work on matter and emitting radiation. (TRACE satellite; Stanford U./Lockheed/NASA.)

Poynting's theorem

Suppose a collection of charges and currents lies entirely within a volume $\mathcal{V}$. If they are released at some point in time, electromagnetic forces will begin to do work on them.

Consider, for instance, the work done *by the forces* on charges and currents in an infinitesimal volume $d\tau$ during a time dt:

$$\begin{aligned} d^2W_{\text{mech.}} &= \mathbf{F}\cdot d\boldsymbol{\ell} = dq\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\mathbf{v}dt \\ &= \mathbf{E}\cdot\mathbf{v}dqdt && \text{i.e. } \mathbf{B} \text{ does no work.} \\ &= \mathbf{E}\cdot\mathbf{J}d\tau dt && \text{using } dq=\rho d\tau,\ \mathbf{J}=\rho\mathbf{v} \end{aligned}$$

Thus, $$\frac{dW_{\text{mech}}}{dt} = \int_{\mathcal{V}} \mathbf{E}\cdot\mathbf{J}d\tau$$

Poynting's theorem (continued)

We've seen this before, of course; it's just another way to write $P = VI$. But we will learn something by elimination of $\boldsymbol{J}$ using Ampère's law:

$$\boldsymbol{J} = \frac{c}{4\pi}\nabla\times\boldsymbol{B} - \frac{1}{4\pi}\frac{\partial\boldsymbol{E}}{\partial t} \quad :$$

$$\frac{dW_{\text{mech.}}}{dt} = \frac{1}{4\pi}\int_{\mathcal{V}} d\tau\left(c\boldsymbol{E}\cdot\left(\nabla\times\boldsymbol{B}\right) - \boldsymbol{E}\cdot\frac{\partial\boldsymbol{E}}{\partial t}\right) \quad .$$

According to Product Rule #6,

$$\boldsymbol{E}\cdot\left(\nabla\times\boldsymbol{B}\right) = -\nabla\cdot\left(\boldsymbol{E}\times\boldsymbol{B}\right) + \boldsymbol{B}\cdot\left(\nabla\times\boldsymbol{E}\right)$$

Use Faraday's law:

$$= -\nabla\cdot\left(\boldsymbol{E}\times\boldsymbol{B}\right) - \frac{1}{c}\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t} \quad .$$

Poynting's theorem (continued)

The time-derivative terms in all of these expressions can be written as

$$\boldsymbol{E}\cdot\frac{\partial \boldsymbol{E}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}\left(\boldsymbol{E}\cdot\boldsymbol{E}\right)=\frac{1}{2}\frac{\partial}{\partial t}E^2 \quad ,$$

and, similarly, $$\boldsymbol{B}\cdot\frac{\partial \boldsymbol{B}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}B^2 \quad .$$

Using the last three results in the expression for *dW/dt* gives us

$$\begin{aligned}\frac{dW_{\text{mech.}}}{dt}&=\frac{1}{4\pi}\int_{\mathcal{V}} d\tau\left(-c\nabla\cdot\left(\boldsymbol{E}\times\boldsymbol{B}\right)-\boldsymbol{B}\cdot\frac{\partial \boldsymbol{B}}{\partial t}-\boldsymbol{E}\cdot\frac{\partial \boldsymbol{E}}{\partial t}\right)\\ &=-\frac{1}{4\pi}\int_{\mathcal{V}} d\tau\left(c\nabla\cdot\left(\boldsymbol{E}\times\boldsymbol{B}\right)+\frac{1}{2}\frac{\partial}{\partial t}\left(B^2+E^2\right)\right) \quad .\end{aligned}$$

Poynting's theorem (continued)

$\mathcal{V}$ is bounded by surface $\mathcal{S}$. Apply the divergence theorem to the first term in the integral, converting it to a surface integral over $\mathcal{S}$:

$$\frac{dW_{\text{mech.}}}{dt} = \frac{1}{4\pi}\int_{\mathcal{V}} d\tau\left(-c\nabla\cdot\left(\boldsymbol{E}\times\boldsymbol{B}\right)-\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t}-\boldsymbol{E}\cdot\frac{\partial\boldsymbol{E}}{\partial t}\right)$$

$$= -\frac{c}{4\pi}\oint_{\mathcal{S}}\left(\boldsymbol{E}\times\boldsymbol{B}\right)\cdot d\boldsymbol{a} - \frac{1}{8\pi}\int_{\mathcal{V}} d\tau\,\frac{\partial}{\partial t}\left(B^2+E^2\right)$$

$$\boxed{= -\frac{c}{4\pi}\oint_{\mathcal{S}}\left(\boldsymbol{E}\times\boldsymbol{B}\right)\cdot d\boldsymbol{a} - \frac{1}{8\pi}\frac{d}{dt}\int_{\mathcal{V}}\left(B^2+E^2\right)d\tau} \quad .$$

Poynting's theorem

In the last step we have used the fact that time is the only variable that survives the integration.

Energy conservation in electrodynamics

Poynting's theorem of course expresses energy conservation.

- We have long known that the energy density stored in electric and magnetic fields – that is , the work required to assemble the charges and currents in the configuration they're in – is
$$u_{EB} = \frac{1}{8\pi}\left(E^2 + B^2\right) \quad .$$
- So the integral of u is evidently the energy within $\mathcal{V}$ stored in the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields, and the term containing it just gives the rate at which the stored energy changes.

Energy conservation in electrodynamics (continued)

- The surface integral term contains a vector field that deserves its own name:

$$\boxed{\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B} \quad .}$$ Poynting vector

- Thus, $$\boxed{\frac{dW_{\text{mech.}}}{dt} = -\oint_{\mathcal{S}} \boldsymbol{S}\cdot d\boldsymbol{a} - \frac{d}{dt}\int_{\mathcal{V}} u_{EB}\,d\tau = -\oint_{\mathcal{S}} \boldsymbol{S}\cdot d\boldsymbol{a} - \frac{dW_{EB}}{dt} \quad .}$$

  This means that the rate at which work is done on the charges and currents by the fields is balanced not just by the rate at which the stored energy decreases, but also by a new term – which, since it involves a flux integral over the surface bounding $\mathcal{V}$, must be the rate at which the fields carry energy out of $\mathcal{V}$.

Energy conservation in electrodynamics (continued)

- Put another way: if the stored energy decreases, not all of the resulting work goes into kinetic energy of the charges – some is **radiated** away. The Poynting vector tells us the energy per unit area and time that is radiated.

Rewrite the last result as

$$\frac{d}{dt}\int_{\mathcal{V}} u_{\mathrm{mech.}}\,d\tau = -\int_{\mathcal{V}} \nabla\cdot\mathbf{S}\,d\tau - \frac{d}{dt}\int_{\mathcal{V}} u_{EB}\,d\tau$$

$$\int_{\mathcal{V}} \frac{\partial u_{\mathrm{mech.}}}{\partial t}\,d\tau = -\int_{\mathcal{V}} \nabla\cdot\mathbf{S}\,d\tau - \int_{\mathcal{V}} \frac{\partial u_{EB}}{\partial t}\,d\tau \quad .$$

We may equate the integrands now, and obtain:

Energy conservation in electrodynamics (continued)

$$\frac{\partial u_{\text{mech.}}}{\partial t} = -\nabla \cdot \boldsymbol{S} - \frac{\partial u_{EB}}{\partial t} \quad , \text{ or}$$

$$\frac{\partial}{\partial t}\left(u_{\text{mech.}} + u_{EB}\right) + \nabla \cdot \boldsymbol{S} = 0 \quad ,$$

an expression that invites comparison with the (charge-current) continuity equation,

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \boldsymbol{J} = 0 \quad .$$

Think of the Poynting vector $\boldsymbol{S}$ therefore as the "current density" of energy.

The Maxwell stress tensor

We won't use tensors very much in this class, and you won't have to cope with this particular tensor after today. But the path to expressions for momentum conservation goes straight through the Maxwell stress tensor, and this tensor does turn out to be a valuable tool in more-advanced courses, so it won't hurt to introduce it here.

Return to the Lorentz force law, and make a new definition:

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right) \quad \Rightarrow \quad \int_{\mathcal{V}} \boldsymbol{\mathcal{F}}\, d\tau = \int_{\mathcal{V}} \rho\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right) d\tau \quad , \text{ where}$$

$$\boldsymbol{\mathcal{F}} = \rho\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right) = \rho\boldsymbol{E} + \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B} \qquad \left(\boldsymbol{J} = \rho\boldsymbol{v}\right)$$

is the force per unit volume exerted by $\boldsymbol{E}$ and $\boldsymbol{B}$.

The Maxwell stress tensor (continued)

Using Gauss's law and Ampère's law, in the forms

$$\rho = \frac{1}{4\pi}\nabla\cdot\boldsymbol{E} \quad , \quad \boldsymbol{J} = \frac{c}{4\pi}\nabla\times\boldsymbol{B} - \frac{1}{4\pi}\frac{\partial\boldsymbol{E}}{\partial t} \quad ,$$

we can eliminate the source terms, in favor of the fields:

$$\boldsymbol{\mathcal{F}} = \frac{1}{4\pi}(\nabla\cdot\boldsymbol{E})\boldsymbol{E} + \frac{1}{4\pi}\left(\nabla\times\boldsymbol{B} - \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}\right)\times\boldsymbol{B} \quad .$$

The very last term can be put into a more useful form by noting that

$$\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{E}\times\boldsymbol{B}) = \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}\times\boldsymbol{B} + \frac{1}{c}\boldsymbol{E}\times\frac{\partial\boldsymbol{B}}{\partial t}$$

Use Faraday's law:

$$= \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}\times\boldsymbol{B} - \boldsymbol{E}\times(\nabla\times\boldsymbol{E}) \quad .$$

The Maxwell stress tensor (continued)

Thus,

$$\mathcal{F}=\frac{1}{4\pi}(\nabla\cdot\boldsymbol{E})\boldsymbol{E}+\frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}+\frac{1}{4\pi}\left(-\frac{\partial}{\partial t}(\boldsymbol{E}\times\boldsymbol{B})-\boldsymbol{E}\times(\nabla\times\boldsymbol{E})\right)$$

$$=\frac{1}{4\pi}\left[(\nabla\cdot\boldsymbol{E})\boldsymbol{E}-\boldsymbol{E}\times(\nabla\times\boldsymbol{E})\right]-\frac{1}{4\pi}\left[\boldsymbol{B}\times(\nabla\times\boldsymbol{B})\right]-\frac{1}{4\pi c}\frac{\partial}{\partial t}(\boldsymbol{E}\times\boldsymbol{B}) \quad .$$

Since $\nabla\cdot\boldsymbol{B}=0$, it changes nothing if we add $(\nabla\cdot\boldsymbol{B})\boldsymbol{B}$ to the second square brackets to make the whole thing look more symmetrical:

$$\mathcal{F}=\frac{1}{4\pi}\left[(\nabla\cdot\boldsymbol{E})\boldsymbol{E}-\boldsymbol{E}\times(\nabla\times\boldsymbol{E})\right]$$

$$+\frac{1}{4\pi}\left[(\nabla\cdot\boldsymbol{B})\boldsymbol{B}-\boldsymbol{B}\times(\nabla\times\boldsymbol{B})\right]-\frac{1}{4\pi c}\frac{\partial}{\partial t}(\boldsymbol{E}\times\boldsymbol{B}) \quad .$$

The Maxwell stress tensor (continued)

We can "simplify" the terms in square brackets if we reintroduce tensor notation. Consider the three Cartesian directions to be represented by $x, y, z = 1, 2, 3$; for example,

$$\boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \quad \text{, or} \quad \boldsymbol{E} = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \quad .$$

Then the x-component of the first term in [] can be written as

$$\left[(\nabla \cdot \boldsymbol{E})\boldsymbol{E} - \boldsymbol{E} \times (\nabla \times \boldsymbol{E})\right]_1 = E_1\left(\frac{\partial E_1}{\partial r_1} + \frac{\partial E_2}{\partial r_2} + \frac{\partial E_3}{\partial r_3}\right)$$
$$- E_2\left(\frac{\partial E_2}{\partial r_1} - \frac{\partial E_1}{\partial r_2}\right) + E_3\left(\frac{\partial E_1}{\partial r_3} - \frac{\partial E_3}{\partial r_1}\right)$$

The Maxwell stress tensor (continued)

It will pay to multiply this out and regroup:

$$
\begin{aligned}
[\ldots]_1 &= \boxed{E_1 \frac{\partial E_1}{\partial r_1}} + \boxed{E_1 \frac{\partial E_2}{\partial r_2}} + \boxed{E_1 \frac{\partial E_3}{\partial r_3}} \\
&\quad - E_2 \frac{\partial E_2}{\partial r_1} + \boxed{E_2 \frac{\partial E_1}{\partial r_2}} + \boxed{E_3 \frac{\partial E_1}{\partial r_3}} - E_3 \frac{\partial E_3}{\partial r_1} \\
&= \boxed{\frac{1}{2} \frac{\partial}{\partial r_1} \left(E_1 \right)^2} + \boxed{\frac{\partial}{\partial r_2} \left(E_1 E_2 \right)} + \boxed{\frac{\partial}{\partial r_3} \left(E_1 E_3 \right)} \\
&\quad - \frac{1}{2} \frac{\partial}{\partial r_1} \left(E_2 \right)^2 - \frac{1}{2} \frac{\partial}{\partial r_1} \left(E_3 \right)^2 \\
&= \frac{\partial}{\partial r_1} \left(E_1 \right)^2 + \frac{\partial}{\partial r_2} \left(E_1 E_2 \right) + \frac{\partial}{\partial r_3} \left(E_1 E_3 \right) - \frac{1}{2} \frac{\partial}{\partial r_1} \left(E_1^2 + E_2^2 + E_3^2 \right)
\end{aligned}
$$

The Maxwell stress tensor (continued)

$$[\ldots]_1 = \frac{\partial}{\partial r_1}\left(E_1\right)^2 + \frac{\partial}{\partial r_2}\left(E_1 E_2\right) + \frac{\partial}{\partial r_3}\left(E_1 E_3\right) - \frac{1}{2}\frac{\partial}{\partial r_1}\left(E_1^2 + E_2^2 + E_3^2\right)$$

$$= \left[\sum_{j=1}^{3}\frac{\partial}{\partial r_j}\left(E_1 E_j\right)\right] - \frac{1}{2}\frac{\partial}{\partial r_1}E^2 = \sum_{j=1}^{3}\frac{\partial}{\partial r_j}\left(E_1 E_j - \frac{1}{2}E^2\delta_{1j}\right) \quad ,$$

where we have reintroduced the Kronecker delta,

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

This is for component 1; for component i, we'd get

$$\left[(\nabla\cdot\boldsymbol{E})\boldsymbol{E} - \boldsymbol{E}\times(\nabla\times\boldsymbol{E})\right]_i = \sum_{j=1}^{3}\frac{\partial}{\partial r_j}\left(E_i E_j - \frac{1}{2}E^2\delta_{ij}\right) \quad .$$

The Maxwell stress tensor (continued)

Similarly,

$$\left[(\nabla\cdot\boldsymbol{B})\boldsymbol{B}-\boldsymbol{B}\times(\nabla\times\boldsymbol{B})\right]_i=\sum_{j=1}^{3}\frac{\partial}{\partial r_j}\left(B_iB_j-\frac{1}{2}B^2\delta_{ij}\right)\quad.$$

Let us now define a nine-component object,

$$\boxed{T_{ij}=\frac{1}{4\pi}\left(E_iE_j+B_iB_j-\frac{1}{2}\left(E^2+B^2\right)\delta_{ij}\right)\quad,}$$

Maxwell stress tensor

with which we can re-write the *i*th component of both of the ugly terms in []:

$$\frac{1}{4\pi}\left[(\nabla\cdot\boldsymbol{E})\boldsymbol{E}-\boldsymbol{E}\times(\nabla\times\boldsymbol{E})\right]_i$$

$$+\frac{1}{4\pi}\left[(\nabla\cdot\boldsymbol{B})\boldsymbol{B}-\boldsymbol{B}\times(\nabla\times\boldsymbol{B})\right]_i=\sum_{j=1}^{3}\frac{\partial}{\partial r_j}T_{ij}\quad.$$

The Maxwell stress tensor (continued)

The best way to envision the sum on the right-hand side is as matrix multiplication:

$$\sum_{j=1}^{3} \frac{\partial}{\partial r_j} T_{ij} = \begin{bmatrix} \frac{\partial}{\partial r_1} & \frac{\partial}{\partial r_2} & \frac{\partial}{\partial r_3} \end{bmatrix} \begin{bmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} \quad .$$

The result is represented by a three-component object; that is, a **vector**. We can use a vector-algebra-like symbolism for this operation, by using $\overleftrightarrow{\boldsymbol{T}}$ to denote the second-rank **tensor** whose nine components are T_{ij}. The sum is represented thus:

$$\sum_{j=1}^{3} \frac{\partial}{\partial r_j} T_{ij} \leftrightarrow \nabla \cdot \overleftrightarrow{\boldsymbol{T}} \quad ,$$

since it now looks so much like a divergence.

The Maxwell stress tensor (continued)

Finally, the force per unit volume becomes, in terms of the Maxwell stress tensor,

$$\mathcal{F} = \frac{1}{4\pi}\left[(\nabla \cdot \boldsymbol{E})\boldsymbol{E} - \boldsymbol{E} \times (\nabla \times \boldsymbol{E})\right]$$

$$+ \frac{1}{4\pi}\left[(\nabla \cdot \boldsymbol{B})\boldsymbol{B} - \boldsymbol{B} \times (\nabla \times \boldsymbol{B})\right] - \frac{1}{4\pi c}\frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B})$$

$$= \nabla \cdot \overleftrightarrow{\boldsymbol{T}} - \frac{\partial}{\partial t}\boldsymbol{g}_{EB} \quad ,$$

where

$$\boxed{\boldsymbol{g}_{EB} \equiv \frac{1}{4\pi c}\boldsymbol{E} \times \boldsymbol{B} = \frac{1}{c^2}\boldsymbol{S} \quad .}$$

Momentum density

Momentum conservation in electrodynamics

The total force on the charges within volume $\mathcal{V}$ can be found by integrating $\boldsymbol{\mathcal{F}}$:

$$\frac{d\boldsymbol{p}_{\text{mech.}}}{dt} = \int_{\mathcal{V}} \boldsymbol{\mathcal{F}} d\tau = \int_{\mathcal{V}} \left(\nabla \cdot \overleftrightarrow{\boldsymbol{T}} - \frac{\partial}{\partial t} \boldsymbol{g}_{EB} \right) d\tau$$

Use divergence theorem:

$$= \oint_{\mathcal{S}} \overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a} - \frac{d}{dt} \int_{\mathcal{V}} \boldsymbol{g}_{EB} d\tau \quad ,$$

We can now identify $\boldsymbol{p}_{EB} \equiv \int_{\mathcal{V}} \boldsymbol{g}_{EB} d\tau$ as the total **momentum** stored in the fields $\boldsymbol{E}$ and $\boldsymbol{B}$, so that $\boldsymbol{g}_{EB} = \frac{1}{4\pi c} \boldsymbol{E} \times \boldsymbol{B}$ is the momentum density in the fields.

Momentum conservation in electrodynamics (continued)

Rearranging slightly, we have

$$\boxed{\frac{d}{dt}\left(\boldsymbol{p}_{\text{mech.}} + \boldsymbol{p}_{EB}\right) = \oint_{\mathcal{S}} \overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a} \quad .}$$

Momentum conservation in electrodynamics

Interpretation:

- Changes in mechanical momentum ($m\boldsymbol{v}$s for all the charges) and momentum stored in fields within $\mathcal{V}$ are caused by the pressure and shear $\left(\overleftrightarrow{\boldsymbol{T}}\right)$ distributed over the boundary surface $\mathcal{S}$, and *vice versa*.
- The terms on the diagonal of $\overleftrightarrow{\boldsymbol{T}}$,

$$\frac{1}{4\pi}\left(E_i^2 + B_i^2 - \frac{1}{2}\left(E^2 + B^2\right)\right) \quad ,$$

represent **pressures** in the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields.

Momentum conservation in electrodynamics (continued)

- The off-diagonal terms, $\frac{1}{4\pi}\left(E_i E_j + B_i B_j\right)$,

  represent **shear** in the fields – the kind of stress that causes strain in a direction different than the stress.

One can also write the momentum-conservation relation in differential form:

$$\frac{d\boldsymbol{p}_{\text{mech.}}}{dt} = \int_{\mathcal{V}} \frac{\partial}{\partial t}\boldsymbol{g}_{\text{mech.}} d\tau = \int_{\mathcal{V}} \left(\nabla \cdot \overleftrightarrow{\boldsymbol{T}} - \frac{\partial}{\partial t}\boldsymbol{g}_{EB}\right) d\tau \quad \text{, or}$$

$$\frac{\partial}{\partial t}\left(\boldsymbol{g}_{\text{mech.}} + \boldsymbol{g}_{EB}\right) - \nabla \cdot \overleftrightarrow{\boldsymbol{T}} = 0 \quad ,$$

where $\boldsymbol{g}_{\text{mech.}}$ is the mechanical (mv-type) momentum per unit volume for the charges within $\mathcal{V}$.

Summary

Electric and magnetic fields can store energy and momentum:

$$u_{EB} = \frac{1}{8\pi}\left(E^2 + B^2\right) \quad , \qquad \text{energy/volume}$$

$$\boldsymbol{g}_{EB} \equiv \frac{1}{4\pi c}\boldsymbol{E}\times\boldsymbol{B} \quad , \qquad \text{momentum/volume}$$

or, for that matter, even angular momentum:

$$\mathcal{L}_{EB} = \boldsymbol{r}\times\boldsymbol{g}_{EB} = \frac{1}{4\pi c}\boldsymbol{r}\times\left(\boldsymbol{E}\times\boldsymbol{B}\right) \quad , \qquad \text{angular momentum/volume}$$

and can **transport** energy (and thus all the other quantities):

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B} \quad . \qquad \text{energy/area/time}$$

Electromagnetic **waves** are the normal method of transport.