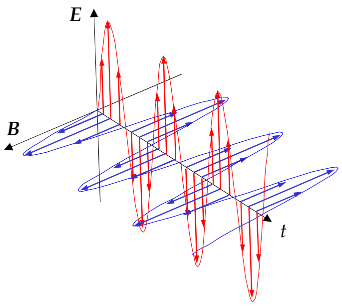
Today in Physics 218: waves

- Electromagnetic waves
- Waves on a string
- The simple solutions to the wave equation
- Sinusoidal waves



Electromagnetic waves

We have already seen that the Maxwell equations can be combined to yield wave equations for the electric and magnetic potentials, if the Lorentz gauge is used. It turns out that they yield wave equations for the fields, too. The Maxwell equations in vacuum with no sources are

$$\nabla\cdot\boldsymbol{E}=0 \qquad \nabla\cdot\boldsymbol{B}=0$$

$$\nabla\times\boldsymbol{E}=-\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t} \qquad \nabla\times\boldsymbol{B}=\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}$$

Take the curl of each of the curl equations:

$$\nabla\times(\nabla\times\boldsymbol{E})=-\frac{1}{c}\frac{\partial}{\partial t}\nabla\times\boldsymbol{B} \quad , \quad \nabla\times(\nabla\times\boldsymbol{B})=\frac{1}{c}\frac{\partial}{\partial t}\nabla\times\boldsymbol{E} \quad .$$

Electromagnetic waves (continued)

Invoke Faraday's and Ampère's law again, and product rule #11:

$$\nabla(\nabla\cdot\boldsymbol{E})-\nabla^2\boldsymbol{E}=-\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}\right) \quad ,$$

$$\nabla(\nabla\cdot\boldsymbol{B})-\nabla^2\boldsymbol{B}=\frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t}\right) \quad ;$$

$$\Rightarrow \quad \nabla^2\boldsymbol{E}=\frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2} \quad , \quad \nabla^2\boldsymbol{B}=\frac{1}{c^2}\frac{\partial^2\boldsymbol{B}}{\partial t^2} \quad .$$

Thus each Cartesian component of each field, in vacuum with no sources, obeys the classical wave equation:

$$\nabla^2 f=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \quad .$$

Electromagnetic waves (continued)

In linear media with no sources, for which the Maxwell equations are

$$\nabla \cdot \boldsymbol{D} = 0 \qquad \nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \times \boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t}$$

and where $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ and $\boldsymbol{B} = \mu \boldsymbol{H}$, the curl equations become

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \quad , \quad \nabla \times \boldsymbol{B} = \frac{\mu\varepsilon}{c}\frac{\partial \boldsymbol{E}}{\partial t} \quad ,$$

so the procedure above yields similar wave equations:

$$\nabla^2 \boldsymbol{E} = \frac{\mu\varepsilon}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \quad , \quad \nabla^2 \boldsymbol{B} = \frac{\mu\varepsilon}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2} \quad .$$

Electromagnetic waves (continued)

The wave equations for linear media have a different wave speed than those for vacuum:

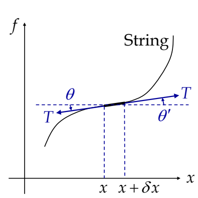
$$v = \frac{c}{\sqrt{\mu\varepsilon}} = \frac{c}{n} \quad ,$$

where $n \equiv \sqrt{\mu\varepsilon}$ is the linear medium's **index of refraction**.

- As you probably already know, the speed that appears in the wave equation is the **phase velocity** of wave solutions, and c is the phase velocity of light in vacuum.
- Many of the important features of electromagnetic waves are displayed graphically by other sorts of waves. In particular, it's worth a close look at waves on a string.

Transverse waves on a string

Consider a string with mass per unit length μ, of very great length, held under a tension T. Denote position along the string as x and a perpendicular displacement of the string from its equilibrium position by f. Displace a very small section δx of the string by a small amount. If displacements are continuous (no kinks), then the forces are:



$$F_x = 0 \quad , \qquad \text{(string doesn't shift or stretch)}$$

$$F_f = T\sin\theta' - T\sin\theta \quad .$$

Transverse waves on a string (continued)

If the string indeed does not stretch or shift horizontally, the angles θ and θ' must be small, in the sense that their sines are equal to their tangents:

$$F_f = T\tan\theta' - T\tan\theta = T\left.\frac{\partial f}{\partial x}\right|_{x+\delta x} - T\left.\frac{\partial f}{\partial x}\right|_{x}$$

Tangent and derivative both describe the slope.

If the displacement is small, we can expand in a Taylor series,

$$\left.\frac{\partial f}{\partial x}\right|_{x+\delta x} = \left.\frac{\partial f}{\partial x}\right|_{x} + \delta x \left.\frac{\partial^2 f}{\partial x^2}\right|_{x} + \ldots$$

and ignore terms of higher order than second:

$$F_f = T\frac{\partial^2 f}{\partial x^2}\delta x = ma = \mu\delta x\frac{\partial^2 f}{\partial t^2} \quad \Rightarrow \quad \frac{\partial^2 f}{\partial x^2} = \left(\frac{\mu}{T}\right)\frac{\partial^2 f}{\partial t^2} \quad .$$

The simple solutions to the wave equation

Thus transverse displacements of the string obey the classical wave equation, with phase velocity $v = \sqrt{T/\mu}$.

Now, there are lots of solutions to the classical wave equation. In fact, *any* functional form is a solution, as long as the function depends upon x and t only through the combination $x \pm vt$; that is,

$$f = g(x \pm vt) = g(z) \quad .$$

$z = x \pm vt =$ **phase**

$g =$ any arbitrary function

This assertion can be proven using nothing more than the chain rule. To wit:

The simple solutions to the wave equation (continued)

$$\frac{\partial f}{\partial x} = \frac{dg}{dz}\frac{\partial z}{\partial x} = \frac{dg}{dz} \qquad \frac{\partial f}{\partial t} = \frac{dg}{dz}\frac{\partial z}{\partial t} = \pm v\frac{dg}{dz}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{dg}{dz} = \frac{d^2 g}{dz^2}\frac{\partial z}{\partial x} = \frac{d^2 g}{dz^2} \qquad \frac{\partial^2 f}{\partial t^2} = \pm v\frac{\partial}{\partial t}\frac{dg}{dz} = \pm v\frac{d^2 g}{dz^2}\frac{\partial z}{\partial t} = v^2\frac{d^2 g}{dz^2}$$

$$\Rightarrow \frac{d^2 g}{dz^2} = \frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \quad \text{(Q.E.D.)}$$

We shall refer to two “simple” solutions, $g(x - vt)$ and $h(x + vt)$.

The simple solutions to the wave equation (continued)

The wave equation is linear, so if g and h are solutions, so is any linear combination of them, such as their sum:

$$f(x,t) = g(x-vt) + h(x+vt) \quad .$$

It is useful to note the form this solution takes at some specific time, say $t = 0$:

$$f(x,0) = g(x) + h(x)$$

$$\frac{\partial f}{\partial t}(x,0) = \frac{dg}{dz}\frac{\partial z_-}{\partial t}\bigg|_{t=0} + \frac{dh}{dz}\frac{\partial z_+}{\partial t}\bigg|_{t=0} \qquad z_\pm = x \pm vt$$

$$\dot{f}(x,0) = \frac{\partial z_-}{\partial t} g'(x) + \frac{\partial z_+}{\partial t} h'(x) \qquad \text{dot} = \frac{\partial}{\partial t}, \text{ prime} = \frac{d}{dz}$$

$$= -vg'(x) + vh'(x) \quad .$$

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The simple solutions to the wave equation (continued)

Integration of both sides of this last expression with respect to x is the same as integration with respect to z, since $t = 0$, and therefore

$$\int_0^x \dot{f}(u,0)\,du = -vg(x) + vh(x) \quad .$$

Solve this and $f(x,0) = g(x) + h(x)$ simultaneously for the two functions, and we get

$$g(x) = \frac{1}{2}\left[f(x,0) - \frac{1}{v}\int_0^x \dot{f}(u,0)\,du\right]$$

$$h(x) = \frac{1}{2}\left[f(x,0) + \frac{1}{v}\int_0^x \dot{f}(u,0)\,du\right]$$

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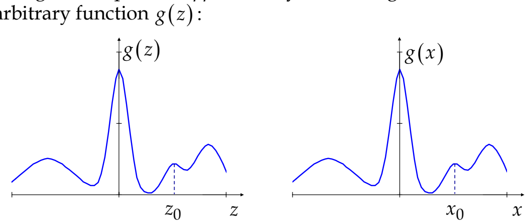
The simple solutions to the wave equation (continued)

Why is it useful to notice this? Because $g(x-vt)$ and $h(x+vt)$ change in simple, but *opposite,* ways as t changes. Consider an arbitrary function $g(z)$:



Plotted as a function of x instead of z, at $t = 0$, it gives the same graph…

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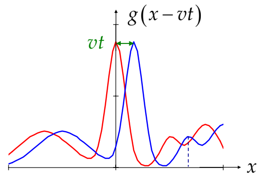
The simple solutions to the wave equation (continued)

…but at a later time t, the value that g has at every x is the same value that the point $x - vt$ had at $t = 0$. That is, the whole pattern has moved to positive x.



$$x = x_0 + vt$$
$$z_0 = x - vt = x_0$$

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The simple solutions to the wave equation (continued)

If you still don't get it: consider one of the peaks in the graphs of g. The peak lies at the same z position z_0 no matter what x and t are, so at time t, its position along the x axis would be given by $z_0 = x_0 = x - vt$, or $x = x_0 + vt$.

So: $g(x - vt)$ is a wave that travels in the +x-direction as time increases. Similarly, $h(x + vt)$ is a wave that travels in the $-x$ direction.

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Sinusoidal waves

Waves of the form
$$f(x,t) = A\cos\left(k(x - vt) + \delta\right) = A\cos(kx - \omega t + \delta) \quad ,$$
where $\omega = vk = 2\pi\nu = 2\pi/\tau$, and
$$k = 2\pi/\lambda \quad ,$$
have special significance, of course, because through Fourier analysis, one may resolve any arbitrary wave $g(x - vt)$ into (Fourier) components of this form.

- ❑ Special quantities: ω is the angular frequency, ν the frequency, and τ the period of oscillation; k the wavenumber and λ the wavelength.

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Sinusoidal waves (continued)

- δ is the phase **delay**. Significance: at t = 0, there is a peak of the sinusoidal wave, located at $x = -\delta/k$, and this peak will be the next one to reach x = 0.
- The wave $f(x,t) = A\cos(kx - \omega t + \delta)$ travels toward $+x$; a sinusoidal wave travelling in the opposite direction will look like $F(x,t) = A\cos(kx + \omega t - \delta)$, so that the next peak to reach x = 0 lies at $x = +\delta/k$ at t = 0 (that is, δ is still a phase *delay*).
- But $\cos\theta = \cos(-\theta)$, so we could just as well write $F(x,t) = A\cos(-kx - \omega t + \delta)$; that is,

$$f(x,t) = A\cos(kx - \omega t + \delta) \quad \Rightarrow$$
$$f(x,t) = A\cos(-kx - \omega t + \delta) \quad \Leftarrow$$

Change the sign of k to reverse the wave's direction.

Sinusoidal waves (continued)

The wave that propagates toward $+x$ can also be written as

$$f(x,t) = \mathrm{Re}\left[Ae^{i(kx-\omega t+\delta)}\right] = \mathrm{Re}\left[\left(Ae^{i\delta}\right)\left(e^{i(kx-\omega t)}\right)\right] \quad .$$

Since manipulation of complex exponentials can be much easier than trig functions, we will often work with the complex wave

$$\tilde{f}(x,t) = \tilde{A}e^{i(kx-\omega t)} \quad ,$$

where $\tilde{A} = Ae^{i\delta}$,

and where A is a real number, with the understanding that the real part must be taken in the end, when physical results are desired.