Today in Physics 218: reflection and transmission

- ❑ Polarization
- ❑ Reflection and transmission of waves on a string
- ❑ Impedance

Rare 360-degree double rainbow over the wild Na Pali coast, Kauai. Photograph by Galen Rowell.

Polarization

The transverse waves we have been discussing are also **linearly polarized**: the direction of propagation and the direction of displacement lie in a single plane; for instance, the wave

$$f(z,t) = \left(\hat{\mathbf{x}}A\cos\theta + \hat{\mathbf{y}}A\sin\theta\right)e^{i(kz-\omega t)}$$

is linearly polarized with its plane of polarization at angle θ from the x axis.

In general a wage could have its two orthogonal components of polarization out of phase, for instance

$$f(z,t) = \left(\hat{\mathbf{x}}A + \hat{\mathbf{y}}Be^{i\phi}\right)e^{i(kz-\omega t)} \quad .$$

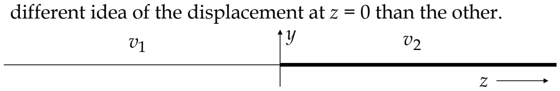
Such a wave is **elliptically polarized**. For the special case $A = B$ and $\phi = \pm\pi/2$, it is **circularly polarized** (see problem 9.8).

Reflection and transmission of waves on a string

Suppose a string is actually made up of two strings with different mass per unit length μ, tied together. How do transverse waves propagate on it?

- ❑ It's still a single string, not shifting or stretching, so both halves have the same tension T.
- ❑ Thus the wave speed $v = \sqrt{T/\mu}$ is different in the two halves.

It's impossible for a single waveform, like $g(z-vt)$, to accommodate both halves; soon one half would have a different idea of the displacement at $z = 0$ than the other.

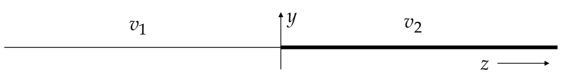


Reflection and transmission (continued)

Easiest way out: consider the string to support multiple waves, which always add up to be continuous and smooth (first derivative continuous) at the junction, $z = 0$.

- ❑ With two waves, continuity could always be guaranteed. With three, both continuity and smoothness could. Let's try three.

Consider the string below. Suppose that someone off at $z = -\infty$ shakes the string up and down at angular frequency ω.



Reflection and transmission (continued)

- ❑ Eventually, all points on the string will oscillate at that **same** angular frequency.
- ❑ Thus $k = v/\omega$ is different on the two sides.
- ❑ Let's suppose the shaking gives rise to a wave that propagates from $z = -\infty$ toward the junction (the incident wave), and that this wave partly continues to propagate on the other half (the transmitted wave), but partly splits off there and propagates back the other way (the reflected wave):

$$\left.\begin{aligned} f_I &= \tilde{A}_I e^{i(k_1 z - \omega t)} \\ f_R &= \tilde{A}_R e^{i(-k_1 z - \omega t)} \end{aligned}\right\} \quad z \le 0 : f = f_I + f_R$$

$$\left. f_T = \tilde{A}_T e^{i(k_2 z - \omega t)} \right\} \quad z \ge 0 : f = f_T$$

Reflection and transmission (continued)

Suppose we know the amplitude of the incident wave. As we've mentioned above, we have enough information to find the corresponding amplitudes of the transmitted and reflected waves, $\tilde{A}_T$ and $\tilde{A}_R$,because

- ❑ the string is continuous through the junction, so
$$f\left(0^-, t\right) = f\left(0^+, t\right).$$
- ❑ the string is smooth through the junction, so
$$\frac{\partial f}{\partial z}\left(0^-, t\right) = \frac{\partial f}{\partial z}\left(0^+, t\right).$$

Two equations, two unknowns.

Reflection and transmission (continued)

Continuity:

$$\left[\tilde{A}_I e^{i(k_1 z-\omega t)} + \tilde{A}_R e^{i(-k_1 z-\omega t)} = \tilde{A}_T e^{i(k_2 z-\omega t)}\right]_{z=0}$$

$$\tilde{A}_I + \tilde{A}_R = \tilde{A}_T$$

Smoothness:

$$\left[ik_1\tilde{A}_I e^{i(k_1 z-\omega t)} - ik_1\tilde{A}_R e^{i(-k_1 z-\omega t)} = ik_2\tilde{A}_T e^{i(k_2 z-\omega t)}\right]_{z=0}$$

$$\tilde{A}_I - \tilde{A}_R = \beta\tilde{A}_T \quad \text{, where}$$

$$\beta = \frac{k_2}{k_1} = \frac{\omega}{v_2}\frac{v_1}{\omega} = \frac{v_1}{v_2} \quad .$$

Reflection and transmission (continued)

Add them directly to get

$$2\tilde{A}_I = (1+\beta)\tilde{A}_T \quad \Rightarrow \quad \tilde{A}_T = \frac{2}{1+\beta}\tilde{A}_I = \frac{2k_1}{k_2+k_1}\tilde{A}_I = \frac{2v_2}{v_1+v_2}\tilde{A}_I \quad .$$

Multiply the first one through by β and subtract them to get

$$(-1+\beta)\tilde{A}_I + (1+\beta)\tilde{A}_R = 0$$

$$\tilde{A}_R = \frac{1-\beta}{1+\beta}\tilde{A}_I = \frac{k_1-k_2}{k_1+k_2}\tilde{A}_I = \frac{v_2-v_1}{v_2+v_1}\tilde{A}_I \quad .$$

Thus

$$f(z,t) = \begin{cases} \tilde{A}_I e^{i(k_1 z-\omega t)} + \dfrac{v_2-v_1}{v_2+v_1}\tilde{A}_I e^{i(-k_1 z-\omega t)} & , z \le 0; \\ \dfrac{2v_2}{v_2+v_1}\tilde{A}_I e^{i(k_2 z-\omega t)} & , z \ge 0. \end{cases}$$

Reflection and transmission (continued)

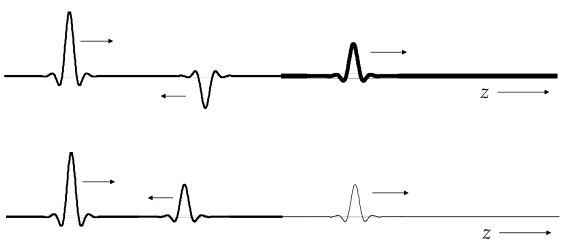
Note that if the $z > 0$ half of the string is lighter (heavier) than the other half, then the wave speed $v = \sqrt{T/\mu}$ is larger (smaller) than it is at $z < 0$. So:

$$\tilde{A}_R = A_R e^{i\delta_R} = \frac{v_2-v_1}{v_2+v_1}A_I e^{i\delta_I}$$

$\Rightarrow \delta_R = \delta_I$ if $v_2 > v_1$, $\delta_R = \delta_I \pm \pi$ otherwise. $\left(\text{Note: } e^{\pm i\pi} = -1.\right)$

- Compared to the incident wave, the reflected one is therefore right-side up (upside down), in the sense that for a given z, the peaks of f_I and the peaks (troughs) of f_R arrive at the same time.
- We say that the incident and reflected waves are in phase $\left(180^\circ \text{ out of phase}\right)$.

Reflection and transmission (continued)



Reflection and transmission, not drawn to scale.

Example: problem !9.7 in the book

Suppose string 2 is embedded in a viscous medium (such as molasses), which imposes a drag force that is proportional to its (transverse) speed: $\Delta F_{\text{drag}} = -\gamma\left(\partial f/\partial t\right)\Delta z$.

(a) *Derive the modified wave equation describing the motion of the string.*

Before adding the molasses, we had $F = T\dfrac{\partial^2 f}{\partial z^2}\Delta z$ (see lecture notes, 26 January). Now,

$$F = T\frac{\partial^2 f}{\partial z^2}\Delta z - \gamma\frac{\partial f}{\partial t}\Delta z = ma = \mu\Delta z\frac{\partial^2 f}{\partial t^2} \quad ;$$

$$\boxed{T\frac{\partial^2 f}{\partial z^2} = \mu\frac{\partial^2 f}{\partial t^2} + \gamma\frac{\partial f}{\partial t}} \quad .$$

Problem !9.7 (continued)

(b) *Solve this equation, assuming that the string oscillates at the incident frequency ω. That is, look for solutions of the form* $\tilde{f}(z,t) = \tilde{F}(z)e^{i\omega t}$.

Actually, $\tilde{f}(z,t) = \tilde{F}(z)e^{-i\omega t}$ is the one you want, to make the wave propagate toward +*z*:

$$Te^{-i\omega t}\frac{d^2\tilde{F}}{dz^2} = \mu\left(-\omega^2\right)\tilde{F}e^{-i\omega t} + \gamma\left(-i\omega\right)\tilde{F}e^{-i\omega t}$$

$$T\frac{d^2\tilde{F}}{dz^2} = -\omega\left(\mu\omega + i\gamma\right)\tilde{F}$$

$$\frac{d^2\tilde{F}}{dz^2} = -\tilde{k}^2\tilde{F} \quad \text{, where } \tilde{k}^2 = \frac{\omega}{T}\left(\mu\omega + i\gamma\right) \quad .$$

Problem !9.7 (continued)

We know the (particular) solution to this equation already:

$$\tilde{F}(z) = \tilde{A}e^{i\tilde{k}z} + \tilde{B}e^{-i\tilde{k}z} \quad .$$

We can go further than this, though, because we can resolve the complex wavenumber $\tilde{k}$ into its real and imaginary parts:

$$\tilde{k} = k + i\kappa$$

$$\tilde{k}^2 = k^2 - \kappa^2 + 2ik\kappa = \frac{\omega}{T}(\mu\omega + i\gamma)$$

$$\therefore 2k\kappa = \frac{\gamma\omega}{T} \quad \Rightarrow \kappa = \frac{\gamma\omega}{2kT} \quad .$$

Problem !9.7 (continued)

$$\therefore k^2 - \kappa^2 = k^2 - \left(\frac{\gamma\omega}{2T}\right)^2 \frac{1}{k^2} = \frac{\mu\omega^2}{T}$$

$$\Rightarrow k^4 - \frac{\mu\omega^2}{T}k^2 - \left(\frac{\gamma\omega}{2T}\right)^2 = 0$$

Solve as a quadratic:

$$k^2 = \frac{1}{2}\left[\frac{\mu\omega^2}{T} \pm \sqrt{\left(\frac{\mu\omega^2}{T}\right)^2 + 4\left(\frac{\gamma\omega}{2T}\right)^2}\right] = \frac{\mu\omega^2}{2T}\left[1 \pm \sqrt{1 + \left(\frac{\gamma}{\mu\omega}\right)^2}\right] \quad .$$

But k is real, so its square must be positive; we need the + root.

Problem !9.7 (continued)

So,

$$k = \omega\sqrt{\frac{\mu}{2T}\left[1 + \sqrt{1 + \left(\frac{\gamma}{\mu\omega}\right)^2}\right]} > 0 \quad ,$$

$$\kappa = \frac{\gamma\omega}{2kT} = \frac{\gamma}{\sqrt{2T\mu\left(1 + \sqrt{1 + \left(\frac{\gamma}{\mu\omega}\right)^2}\right)}} > 0 \quad .$$

Plug back into the original solution:

$$\tilde{F}(z) = \tilde{A}e^{ikz}e^{-\kappa z} + \tilde{B}e^{-ikz}e^{\kappa z} \quad .$$

The second term increases exponentially with increasing z (unphysical!), so we must have $B = 0$.

Problem !9.7 (continued)

Thus $\boxed{\tilde{f}(z,t)=\tilde{A}e^{-\kappa z}e^{i(kz-\omega t)}\quad,}$

with k and κ as given above. (Take real part for actual string displacement.)

(c) *Show that the waves are* ***attenuated*** *(that is, their amplitude decreases with increasing z). Find the characteristic penetration distance, at which the amplitude is 1/e of its original value.*

That's obvious in the exponential with the real argument:

$$\frac{1}{e}=e^{-\kappa z_0}\quad\Rightarrow\quad z_0=\frac{1}{\kappa}=\frac{1}{\gamma}\sqrt{2T\mu\left(1+\sqrt{1+\left(\frac{\gamma}{\mu\omega}\right)^2}\right)}\quad .$$

Problem !9.7 (continued)

(d) *If a wave of amplitude* A_I, *phase* δ_I, *and frequency* ω *is incident from the left (string 1), find the reflected wave's amplitude and phase.*

We can simply use our previous result, with $k_2=k+i\kappa$:

$$\tilde{A}_R=\frac{k_1-k_2}{k_1+k_2}\tilde{A}_I=\frac{k_1-k-i\kappa}{k_1+k+i\kappa}\tilde{A}_I$$

$$\left|\frac{\tilde{A}_R}{\tilde{A}_I}\right|^2=\left(\frac{k_1-k-i\kappa}{k_1+k+i\kappa}\right)\left(\frac{k_1-k+i\kappa}{k_1+k-i\kappa}\right)=\frac{(k_1-k)^2+\kappa^2}{(k_1+k)^2+\kappa^2}$$

$$\boxed{\frac{A_R}{A_I}=\sqrt{\frac{(k_1-k)^2+\kappa^2}{(k_1+k)^2+\kappa^2}}}$$

where $k_1=\omega/v_1=\sqrt{\mu_1/T}$, k and κ as above.

Problem !9.7 (continued)

And now for the phase:

$$\tan(\delta_R-\delta_I)=\operatorname{Im}\left(\frac{\tilde{A}_R}{\tilde{A}_I}\right)\Big/\operatorname{Re}\left(\frac{\tilde{A}_R}{\tilde{A}_I}\right)$$

$$\frac{\tilde{A}_R}{\tilde{A}_I}=\left(\frac{k_1-k-i\kappa}{k_1+k+i\kappa}\right)=\left(\frac{k_1-k-i\kappa}{k_1+k+i\kappa}\right)\left(\frac{k_1+k-i\kappa}{k_1+k-i\kappa}\right)$$

$$=\frac{k_1^2+k_1k-ik_1\kappa-kk_1-k^2+ik\kappa-ik_1\kappa-ik\kappa-\kappa^2}{(k_1+k)^2+\kappa^2}$$

$$=\frac{k_1^2-k^2-\kappa^2-2ik_1\kappa}{(k_1+k)^2+\kappa^2}$$

Problem !9.7 (continued)

so, finally,

$$\tan\left(\delta_R - \delta_I\right) = \frac{-2k_1\kappa}{k_1^2 - k^2 - \kappa^2} \quad ,$$

$$\boxed{\delta_R - \delta_I = \arctan\left(\frac{-2k_1\kappa}{k_1^2 - k^2 - \kappa^2}\right)} \quad .$$

- Note that, throughout, we only get amplitudes and phases relative to those of the incident wave.

Reflection, transmission and impedance

Consider a simple transverse wave again:

$$f(z,t) = Ae^{i(kx-\omega t)} \quad ,$$

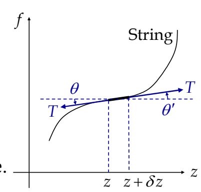
for which $\dfrac{\partial f}{\partial z} = ikAe^{i(kx-\omega t)}$, $\dfrac{\partial f}{\partial t} = -i\omega Ae^{i(kx-\omega t)}$.

Consider again the force diagram for the string:

$$F_f(z) = T\sin\theta \cong T\tan\theta = T\frac{\partial f}{\partial z}$$

$$= T\left(-\frac{k}{\omega}\right)\frac{\partial f}{\partial t} = -\frac{T}{v}\frac{\partial f}{\partial t} \equiv -Z\frac{\partial f}{\partial t} \quad .$$

$Z = T/v = \sqrt{T\mu}$ is called the **impedance**.



Reflection, transmission and impedance (continued)

In these terms, the two halves of our string have different impedances. We can case the reflected and transmitted wave amplitudes in this form:

$$\tilde{A}_R = \frac{v_2 - v_1}{v_2 + v_1}\tilde{A}_I = \frac{\dfrac{T}{Z_2} - \dfrac{T}{Z_1}}{\dfrac{T}{Z_2} + \dfrac{T}{Z_1}}\tilde{A}_I = \frac{Z_1 - Z_2}{Z_1 + Z_2}\tilde{A}_I \quad .$$

Similarly, $\tilde{A}_T = \dfrac{2Z_1}{Z_1 + Z_2}\tilde{A}_I$.

Changes in impedance are associated with reflection. What does the impedance "mean"? Consider the power carried by the wave past some point z:

Reflection, transmission and impedance (continued)

$$P = \frac{d}{dt}\left(\boldsymbol{F}\cdot\boldsymbol{\ell}\right) = \boldsymbol{F}\cdot\frac{d\boldsymbol{\ell}}{dt} \qquad \text{(since the wave is transverse)}$$

$$= -F_f\frac{\partial f}{\partial t} = -F_f\left(-v\frac{\partial f}{\partial z}\right) = -\frac{v}{T}F_f\left(-T\frac{\partial f}{\partial z}\right) = \frac{1}{Z}F_f^2 \quad ,$$

or

$$= \left(T\frac{\partial f}{\partial z}\right)\frac{\partial f}{\partial t} = \frac{T}{v}\left(\frac{\partial f}{\partial t}\right)^2 = Z\left(\frac{\partial f}{\partial t}\right)^2 \quad .$$

Compare $P = \frac{1}{Z}F_f^2 = Z\left(\frac{\partial f}{\partial t}\right)^2$ to some familiar electrical quantities:

$$P = \frac{1}{R}V^2 = RI^2 \quad (\text{DC}), \quad P = \mathrm{Re}\left(\frac{1}{Z}V^2\right) = \mathrm{Re}\left(ZI^2\right) \quad (\text{AC}).$$