Today in Physics 218: back to electromagnetic waves

- Impedance
- Plane electro-magnetic waves
- Energy and momentum in plane electro-magnetic waves
- Radiation pressure

Artist's conception of a "solar sail:" a spacecraft propelled by solar radiation pressure. (Benjamin Diedrich, Caltech.)

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Reflection, transmission and impedance

Consider a simple transverse wave again:

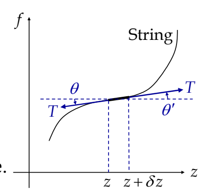
$$f(z,t) = Ae^{i(kx-\omega t)} \quad ,$$

for which $\dfrac{\partial f}{\partial z} = ikAe^{i(kx-\omega t)}$, $\dfrac{\partial f}{\partial t} = -i\omega Ae^{i(kx-\omega t)}$.

Consider again the force diagram for the string:

$$F_f(z) = T\sin\theta \cong T\tan\theta = T\frac{\partial f}{\partial z}$$

$$= T\left(-\frac{k}{\omega}\right)\frac{\partial f}{\partial t} = -\frac{T}{v}\frac{\partial f}{\partial t} \equiv -Z\frac{\partial f}{\partial t} \quad .$$



$Z = T/v = \sqrt{T\mu}$ is called the **impedance**.

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Reflection, transmission and impedance (continued)

In these terms, the two halves of our string have different impedances. We can case the reflected and transmitted wave amplitudes in this form:

$$\tilde{A}_R = \frac{v_2 - v_1}{v_2 + v_1}\tilde{A}_I = \frac{\dfrac{T}{Z_2} - \dfrac{T}{Z_1}}{\dfrac{T}{Z_2} + \dfrac{T}{Z_1}}\tilde{A}_I = \frac{Z_1 - Z_2}{Z_1 + Z_2}\tilde{A}_I \quad .$$

Similarly, $\tilde{A}_T = \dfrac{2Z_1}{Z_1 + Z_2}\tilde{A}_I$.

Changes in impedance are associated with reflection. What does the impedance "mean"? Consider the power carried by the wave past some point z:

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Reflection, transmission and impedance (continued)

$$P = \frac{d}{dt}(\boldsymbol{F}\cdot\boldsymbol{\ell}) = \boldsymbol{F}\cdot\frac{d\boldsymbol{\ell}}{dt} \qquad \text{(since the wave is transverse)}$$

$$= -F_f\frac{\partial f}{\partial t} = -F_f\left(-v\frac{\partial f}{\partial z}\right) = -\frac{v}{T}F_f\left(-T\frac{\partial f}{\partial z}\right) = \frac{1}{Z}F_f^2 \quad ,$$

or

$$= \left(T\frac{\partial f}{\partial z}\right)\frac{\partial f}{\partial t} = \frac{T}{v}\left(\frac{\partial f}{\partial t}\right)^2 = Z\left(\frac{\partial f}{\partial t}\right)^2 \quad .$$

Compare $P = \frac{1}{Z}F_f^2 = Z\left(\frac{\partial f}{\partial t}\right)^2$ to some familiar electrical quantities:

$$P = \frac{1}{R}V^2 = RI^2 \quad \text{(DC)}, \quad P = \mathrm{Re}\left(\frac{1}{Z}V^2\right) = \mathrm{Re}\left(ZI^2\right) \quad \text{(AC)}.$$

Back to electromagnetic waves

With strings providing an introduction, we will now return to electromagnetic waves, which you will recall involve fields that, in vacuum, obey classical wave equations:

$$\nabla^2\boldsymbol{E} = \frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2} \quad , \quad \nabla^2\boldsymbol{B} = \frac{1}{c^2}\frac{\partial^2\boldsymbol{B}}{\partial t^2}.$$

As with strings, the simplest solutions are the sinusoidal ones, called plane waves:

$$\tilde{\boldsymbol{E}}(\boldsymbol{r},t) = \tilde{\boldsymbol{E}}_0 e^{-(kz-\omega t)} \quad , \quad \tilde{\boldsymbol{B}}(\boldsymbol{r},t) = \tilde{\boldsymbol{B}}_0 e^{-(kz-\omega t)} \quad ,$$

where $\tilde{\boldsymbol{E}}_0$ and $\tilde{\boldsymbol{B}}_0$ are complex constants. First, a few remarks about the nature of electromagnetic plane waves in vacuum are in order:

Relationship of *E* and *B*

- Because the divergences of the fields are both zero in vacuum, we have, for instance,

$$\nabla\cdot\tilde{\boldsymbol{E}} = \frac{\partial}{\partial z}\tilde{E}_{0z}e^{i(kz-\omega t)} \qquad \text{No } x \text{ or } y \text{ dependence}$$

$$= ik\tilde{E}_{0z}e^{i(kz-\omega t)} = 0 \quad \Rightarrow \quad \tilde{E}_{0z} = 0.$$

  Similarly, $\tilde{B}_{0z} = 0$. Electromagnetic waves in vacuum have no field components along their direction of propagation; they're transverse, just like waves on a string.
- We learn something from the curls, too:

$$\nabla\times\tilde{\boldsymbol{E}} = -\frac{\partial\tilde{E}_y}{\partial z}\hat{\boldsymbol{x}} + \frac{\partial\tilde{E}_x}{\partial z}\hat{\boldsymbol{y}} \qquad \text{All other terms vanish}$$

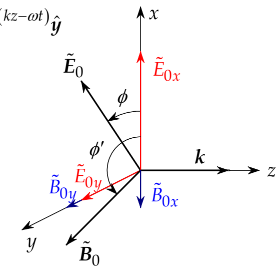
Relationship of *E* and *B* (continued)

$$\nabla \times \tilde{\boldsymbol{E}} = -ik\tilde{E}_{0y}e^{i(kz-\omega t)}\hat{\boldsymbol{x}} + ik\tilde{E}_{0x}e^{i(kz-\omega t)}\hat{\boldsymbol{y}}$$
$$= -\frac{1}{c}\frac{\partial \tilde{\boldsymbol{B}}}{\partial t} = \frac{i\omega}{c}\tilde{\boldsymbol{B}}_0 e^{i(kz-\omega t)}$$
$$= \frac{i\omega}{c}\tilde{B}_{0x}e^{i(kz-\omega t)}\hat{\boldsymbol{x}}$$
$$+ ik\tilde{B}_{0y}e^{i(kz-\omega t)}\hat{\boldsymbol{y}}.$$

But $k = \omega/c$, so

$$\tilde{E}_{0x} = \tilde{B}_{0y} \quad , \quad \tilde{E}_{0y} = -\tilde{B}_{0x} \quad ,$$

which has interesting implications for the relative magnitude, phase, and orientation of the fields:



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Relationship of *E* and *B* (continued)

the field amplitudes have the **same magnitude**, as

$$\left|\tilde{E}_0\right| = \sqrt{\tilde{E}_{0x}^2 + \tilde{E}_{0y}^2} = \sqrt{\tilde{B}_{0y}^2 + \tilde{B}_{0x}^2} = \left|\tilde{B}_0\right| \quad ;$$

are **perpendicular** to one another, as

$$\tan\phi = \frac{\tilde{E}_{0y}}{\tilde{E}_{0x}} \quad , \quad \tan\phi' = \frac{\tilde{B}_{0y}}{\tilde{B}_{0x}} = -\frac{\tilde{E}_{0x}}{\tilde{E}_{0y}} = -\tan\phi = \tan\left(\phi + \frac{\pi}{2}\right);$$

and are **in phase**, as we see by combining the last three results:

$$\tilde{\boldsymbol{B}}_0 = \hat{\boldsymbol{z}} \times \tilde{\boldsymbol{E}}_0 \quad .$$

In MKS, we'd get $\left|\tilde{\boldsymbol{E}}_0\right| = c\left|\tilde{\boldsymbol{B}}_0\right|$, but the fields would still be perpendicular and in phase.

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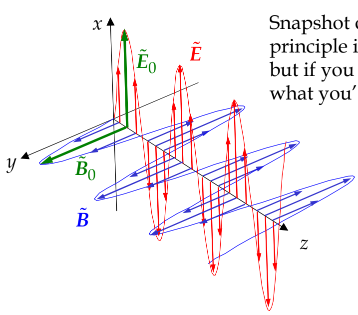
Relationship of *E* and *B* (continued)



Snapshot of the fields – in principle impossible to do, but if you could this is what you'd see)

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Energy and momentum in electromagnetic waves

More generally, if the wave propagates in a direction other than z (i.e. the wavenumber is a vector, $\boldsymbol{k}$), then

$$\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \quad , \quad \tilde{\boldsymbol{B}} = \hat{\boldsymbol{k}} \times \tilde{\boldsymbol{E}} \quad .$$

Taking $\boldsymbol{E}$ and $\boldsymbol{B}$ to be the real parts of the plane-wave fields $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{B}}$, we get the following for the energy density, and energy flux:

$$u = \frac{1}{8\pi}\left(E^2 + B^2\right) = \boxed{\frac{E^2}{4\pi} = \frac{B^2}{4\pi}} \qquad = \varepsilon_0 E^2 = \frac{B^2}{\mu_0} \text{ in MKS.}$$

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B} = \frac{c}{4\pi}\boldsymbol{E}\times\left(\hat{\boldsymbol{k}}\times\boldsymbol{E}\right) \qquad \text{Use Product Rule \#2:}$$

$$= \frac{c}{4\pi}\hat{\boldsymbol{k}}\left(\boldsymbol{E}\cdot\boldsymbol{E}\right) - \frac{c}{4\pi}\boldsymbol{E}\left(\hat{\boldsymbol{k}}\cdot\boldsymbol{E}\right) = \boxed{\frac{cE^2}{4\pi}\hat{\boldsymbol{k}} = cu\hat{\boldsymbol{k}}} \quad .$$

Last one same in MKS.

Energy and momentum in electromagnetic waves (continued)

For the momentum density,

$$\boldsymbol{g} = \frac{\boldsymbol{S}}{c^2} = \boxed{\frac{E^2}{4\pi c}\hat{\boldsymbol{k}} = \frac{u}{c}\hat{\boldsymbol{k}}} \quad .$$

Last one same in MKS.

So, unsurprisingly, an electromagnetic wave carries energy and momentum in its direction of motion.

In the case of light, the periods of oscillation are so short that time averages of these quantities is useful. Recall that the average over a period for the square of a cosine or a sine is

$$\left\langle \cos^2 \omega t \right\rangle \equiv \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \cos^2 \omega t\, dt = \frac{1}{2} = \left\langle \sin^2 \omega t \right\rangle \quad .$$

Flashback (part e, problem 7.54)

Remember the orthonormality of sines and cosines?

$$\int_0^{2\pi} \cos mx \cos nx\, dx = \int_0^{2\pi} \sin mx \sin nx\, dx = \pi\delta_{mn} \quad ,$$

$$\int_0^{2\pi} \cos mx \sin nx\, dx = 0 \quad \text{, so}$$

$$\left\langle \cos^2 \omega t \right\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \cos^2 \omega t\, dt = \frac{1}{2\pi}\int_0^{2\pi} \cos^2 x\, dx = \frac{1}{2} \quad ,$$

and, similarly,

$$\left\langle \sin^2 \omega t \right\rangle = \frac{1}{2} \quad \text{and} \quad \left\langle \cos \omega t \sin \omega t \right\rangle = 0 \quad .$$

Energy and momentum in electromagnetic waves (continued)

Thus,

$$\langle u \rangle = \left\langle \frac{E^2}{4\pi} \right\rangle = \frac{E_0^2}{4\pi}\left\langle \cos^2\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right)\right\rangle = \frac{E_0^2}{8\pi} \quad .$$

Similarly,

$$\langle \mathbf{S} \rangle = \left\langle cu\hat{\mathbf{k}} \right\rangle = \frac{cE_0^2}{8\pi}\hat{\mathbf{k}} \equiv I\hat{\mathbf{k}} \quad , \qquad I = \text{ intensity}$$

$$\langle \mathbf{g} \rangle = \left\langle \frac{u}{c}\hat{\mathbf{k}} \right\rangle = \frac{E_0^2}{8\pi c}\hat{\mathbf{k}} \quad .$$

Radiation pressure

When light falls encounters a surface bounding a medium, it can be absorbed or reflect, in either case imparting its momentum to the medium and thus exerting a pressure. The momentum per unit area crossing an area A during a time Δt is

$$\Delta p = \langle g \rangle c \Delta t \quad .$$

If the medium is a perfect absorber, the pressure is therefore

$$P_{\text{abs}} = \frac{1}{A}\frac{dp}{dt} = \frac{1}{A}\frac{\Delta p}{\Delta t} = c\langle g \rangle \boxed{= \frac{E_0^2}{8\pi} = u = \frac{I}{c}} \quad .$$

If the medium's surface is a perfect reflector, the light rebounds elastically and the pressure is twice as high:

$$P_{\text{abs}} = 2P_{\text{ref}} = \frac{E_0^2}{4\pi} \quad .$$

Example: problem 9.10

The intensity of sunlight hitting the Earth is about 1300 W/m². If sunlight strikes a perfect absorber, what pressure does it exert? How about a perfect reflector? What fraction of atmospheric pressure does this amount to?

In cgs units, that's 1.3×10^6 erg sec^{-1} cm^{-2} :

$$P_{\text{abs}} = \frac{I}{c} = 4.3\times10^{-5} \text{ dyne cm}^{-2} \quad .$$

It's twice that much for a reflector. Either way it's not much compared to atmospheric pressure,

$$P_{\text{atm}} = 1.013\times10^6 \text{ dyne cm}^{-2} ; \frac{P_{\text{abs}}}{P_{\text{atm}}} = 4.3\times10^{-11}.$$