Today in Physics 218: electromagnetic waves in linear media

- Their energy and momentum
- Their reflectance and transmission, for normal incidence
- Their polarization

Sunrise over Victoria Falls, Zambezi River. (Photo by Galen Rowell.)

Energy and momentum in electromagnetic waves in linear media

Unlike the wave equations, we can't get the correct results for energy- and momentum-related quantities just by replacing c with $c/\sqrt{\mu\varepsilon}$. Reverting to complex amplitudes and plane waves, we can obtain from the linear-media form of Faraday's law,

$$\nabla\times\tilde{\boldsymbol{E}} = -\frac{\partial \tilde{E}_y}{\partial z}\hat{\boldsymbol{x}} + \frac{\partial \tilde{E}_x}{\partial z}\hat{\boldsymbol{y}} = -ik\tilde{E}_{0y}e^{i(kz-\omega t)}\hat{\boldsymbol{x}} + ik\tilde{E}_{0x}e^{i(kz-\omega t)}\hat{\boldsymbol{y}}$$

$$= -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = \frac{i\omega}{c}\left(\tilde{B}_{0x}\hat{\boldsymbol{x}} + \tilde{B}_{0y}\hat{\boldsymbol{y}}\right)e^{i(kz-\omega t)} \quad .$$

Now, $k = \omega/v = \omega\sqrt{\mu\varepsilon}/c$, so

$$\sqrt{\mu\varepsilon}\tilde{E}_{0x} = \tilde{B}_{0y} \quad , \quad \sqrt{\mu\varepsilon}\tilde{E}_{0y} = -\tilde{B}_{0x} \quad \Rightarrow \quad \boxed{\boldsymbol{B} = \sqrt{\mu\varepsilon}\hat{\boldsymbol{z}}\times\boldsymbol{E}} \quad .$$

Energy and momentum in electromagnetic waves in linear media (continued)

To get the correct form for energy density in such waves, we need to go through some of the derivation of Poynting's theorem again (see lecture notes for 23 January):

$$\boldsymbol{J} = \frac{c}{4\pi}\nabla\times\boldsymbol{H} - \frac{1}{4\pi}\frac{\partial \boldsymbol{D}}{\partial t}$$

$$\frac{dW_{\text{mech}}}{dt} = \int_{\mathcal{V}} \boldsymbol{E}\cdot\boldsymbol{J}\,d\tau = \frac{1}{4\pi}\int_{\mathcal{V}} d\tau\left(c\boldsymbol{E}\cdot(\nabla\times\boldsymbol{H}) - \boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t}\right)$$

Use product rule #6

$$= \frac{1}{4\pi}\int_{\mathcal{V}} d\tau\left(-c\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) - \boldsymbol{H}\cdot\frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t}\right)$$

But $\boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t} = \frac{\varepsilon}{2}\frac{\partial}{\partial t}(\boldsymbol{E}\cdot\boldsymbol{E}) = \frac{\varepsilon}{2}\frac{\partial}{\partial t}E^2 = \frac{1}{2}\frac{\partial}{\partial t}(\boldsymbol{E}\cdot\boldsymbol{D})$.

Energy and momentum in electromagnetic waves in linear media (continued)

Similarly, $\boldsymbol{H}\cdot\frac{\partial \boldsymbol{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\boldsymbol{B}\cdot\boldsymbol{H})$, so

$$\frac{dW_{\text{mech.}}}{dt} = \frac{1}{4\pi}\int_{\mathcal{V}} d\tau\left(-c\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) - \frac{1}{2}\frac{\partial}{\partial t}(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{B}\cdot\boldsymbol{H})\right)$$

$$= -\frac{c}{4\pi}\oint_{\mathcal{S}}(\boldsymbol{E}\times\boldsymbol{H})\cdot d\boldsymbol{a} - \frac{1}{8\pi}\frac{d}{dt}\int_{\mathcal{V}}(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{B}\cdot\boldsymbol{H})\,d\tau$$

$$= -\oint_{\mathcal{S}}\boldsymbol{S}\cdot d\boldsymbol{a} - \frac{1}{8\pi}\frac{d}{dt}\int_{\mathcal{V}} u\,d\tau \quad \text{, where}$$

$$\boxed{\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{H} = \frac{c}{4\pi\mu}\boldsymbol{E}\times\boldsymbol{B}} \;,$$

Energy and momentum in electromagnetic waves in linear media (continued)

and $\boxed{u = \frac{1}{8\pi}(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{B}\cdot\boldsymbol{H}) = \frac{1}{8\pi}\left(\varepsilon E^2 + \frac{1}{\mu}B^2\right).}$

To get the momentum density properly in linear media, we would similarly have to retrace our steps through the derivation of momentum conservation that involved the Maxwell stress tensor. That's not hard, but not very illuminating, so we'll take a short cut to the answer:

$$\boxed{\boldsymbol{g} = \varepsilon\mu\boldsymbol{S} = \frac{\varepsilon\mu}{4\pi c}\boldsymbol{E}\times\boldsymbol{H} = \frac{\varepsilon}{4\pi c}\boldsymbol{E}\times\boldsymbol{B} \quad .}$$

This is one of the few instances in which we could have saved ourselves writing by starting with the MKS expressions.

Reflection and transmission of electromagnetic waves

Consider plane electromagnetic waves in a universe consisting of two **non-conducting** linear media with different permittivity and permeability. The propagation speed is different, and thus reflection can occur at their interface.

- For strings, we worked out the amplitudes of reflected and transmitted waves by applying the boundary conditions implied by string continuity and smoothness.
- For fields, we have other boundary conditions (c.f. the lecture notes for 16 January); labelling the media 1 and 2,

$$\varepsilon_1 E_{\perp,1} - \varepsilon_2 E_{\perp,2} = 4\pi\sigma_f = 0 \qquad B_{\perp,1} - B_{\perp,2} = 0$$

$$E_{\parallel,1} - E_{\parallel,2} = 0 \qquad \frac{1}{\mu_1}B_{\parallel,1} - \frac{1}{\mu_2}B_{\parallel,2} = \frac{4\pi}{c}\left|\boldsymbol{K}_f\times\hat{\boldsymbol{n}}\right| = 0$$

Reflection and transmission of electromagnetic waves (continued)

Suppose an infinite plane (z = 0) separates the two media, and that a plane wave with $\tilde{\boldsymbol{E}}$ polarized in the x direction is incident along z (that is, incident normally). What are the reflected and transmitted amplitudes of $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{B}}$?
See next slide, and write

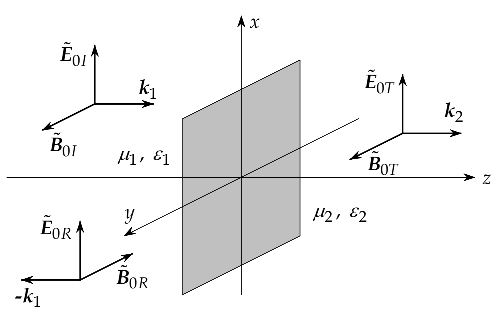
$$\tilde{\boldsymbol{E}}_I = \hat{\boldsymbol{x}} E_{0I} e^{i(k_1 z - \omega t)} \qquad \tilde{\boldsymbol{B}}_I = \hat{\boldsymbol{y}} B_{0I} e^{i(k_1 z - \omega t)} = \hat{\boldsymbol{y}} \sqrt{\mu_1 \varepsilon_1} E_{0I} e^{i(k_1 z - \omega t)}$$

$$\tilde{\boldsymbol{E}}_R = \hat{\boldsymbol{x}} E_{0R} e^{i(-k_1 z - \omega t)} \qquad \tilde{\boldsymbol{B}}_R = -\hat{\boldsymbol{y}} \sqrt{\mu_1 \varepsilon_1} E_{0R} e^{i(-k_1 z - \omega t)}$$

$$\tilde{\boldsymbol{E}}_T = \hat{\boldsymbol{x}} E_{0T} e^{i(k_2 z - \omega t)} \qquad \tilde{\boldsymbol{B}}_T = \hat{\boldsymbol{y}} \sqrt{\mu_2 \varepsilon_2} E_{0T} e^{i(k_2 z - \omega t)}$$

Note that $\tilde{\boldsymbol{B}}_{0R}$ must point along $-x$, because $\boldsymbol{B} = \sqrt{\mu\varepsilon}\,\hat{\boldsymbol{k}} \times \boldsymbol{E}$.

Reflection and transmission of electromagnetic waves (continued)



Reflection and transmission of electromagnetic waves (continued)

Since there are no components of either field perpendicular to the surface, only the parallel-component continuity condition is helpful. At z = 0, continuity of $\boldsymbol{E}_{\|}$ gives us

$$E_{Ix}(0,t) + E_{Rx}(0,t) = E_{Tx}(0,t)$$
$$\tilde{E}_{0I} e^{-i\omega t} + \tilde{E}_{0R} e^{-i\omega t} = \tilde{E}_{0T} e^{-i\omega t} \quad .$$

Continuity of $\boldsymbol{H}_{\|}$ gives us

$$H_{Iy}(0,t) + H_{Ry}(0,t) = H_{Ty}(0,t)$$
$$\frac{1}{\mu_1}\sqrt{\mu_1\varepsilon_1}\,\tilde{E}_{0I} e^{-i\omega t} - \frac{1}{\mu_1}\sqrt{\mu_1\varepsilon_1}\,\tilde{E}_{0R} e^{-i\omega t} = \frac{1}{\mu_2}\sqrt{\mu_2\varepsilon_2}\,\tilde{E}_{0T} e^{-i\omega t} \quad .$$

Reflection and transmission of electromagnetic waves (continued)

Cancel common factors, divide the $\boldsymbol{H}$ equations through by $\sqrt{\varepsilon_1/\mu_1}$, and define $\beta = \sqrt{\varepsilon_2\mu_1/\varepsilon_1\mu_2}$, to get

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \quad ,$$
$$\tilde{E}_{0I} - \tilde{E}_{0R} = \beta\tilde{E}_{0T} \quad .$$

Hey, that's exactly like reflection in a vibrating two-part string (lecture notes, 28 January); the solution is

$$\boxed{\begin{aligned}\tilde{E}_{0T} &= \frac{2}{1+\beta}\tilde{E}_{0I} \quad , \\ \tilde{E}_{0R} &= \frac{1-\beta}{1+\beta}\tilde{E}_{0I} \quad .\end{aligned}}$$

Reflection and transmission of electromagnetic waves (continued)

Only one kind of magnetic material has relative permeability much different from unity: ferromagnets. But ferromagnets are all conductors, so since we are considering non-conductors at the moment we can take $\mu = 1$. Then $\beta = \sqrt{\varepsilon_2/\varepsilon_1} = n_2/n_1 = v_1/v_2$, so

$$\boxed{\begin{aligned}\tilde{E}_{0T} &= \frac{2v_2}{v_2+v_1}\tilde{E}_{0I} = \frac{2n_1}{n_1+n_2}\tilde{E}_{0I} \quad , \\ \tilde{E}_{0R} &= \frac{v_2-v_1}{v_2+v_1}\tilde{E}_{0I} = \frac{n_1-n_2}{n_1+n_2}\tilde{E}_{0I} \quad .\end{aligned}} \quad (\mu = 1)$$

Just as for strings, the reflected wave is in phase with incident if $v_2 > v_1$, and 180° out of phase (upside down) if not.

Reflection and transmission of electromagnetic waves (continued)

That's for electric field amplitude; what about intensity? If $\mu = 1$, then

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B} = \frac{c\sqrt{\varepsilon}}{4\pi}E^2\hat{\boldsymbol{k}} \quad \Rightarrow \quad I = \frac{c\sqrt{\varepsilon}}{8\pi}E^2 = \frac{cn}{8\pi}E^2 \quad ,$$

so

$$T = \frac{I_T}{I_I} = \frac{n_2}{n_1}\frac{E_{0T}^2}{E_{0I}^2} = \frac{n_2}{n_1}\left(\frac{2n_1}{n_1+n_2}\right)^2 = \boxed{\frac{4n_1n_2}{\left(n_1+n_2\right)^2}} \quad ,$$

$$R = \frac{I_R}{I_I} = \frac{E_{0R}^2}{E_{0I}^2} = \boxed{\left(\frac{n_1-n_2}{n_1+n_2}\right)^2} \quad .$$

Of course (energy conservation),

$$T + R = \frac{4n_1n_2}{\left(n_1+n_2\right)^2} + \left(\frac{n_1-n_2}{n_1+n_2}\right)^2 = \left(\frac{n_1+n_2}{n_1+n_2}\right)^2 = 1 \quad .$$

Polarization of transmitted and reflected light

An interesting point is raised by problem 9.14:
In writing 9.76 and 9.77, I tacitly assumed that the reflected and transmitted waves must have the same polarization as the incident wave – along the x *direction. Prove that this* must *be so. [Hint: let the polarization vectors of the transmitted and reflected waves be*

$$\hat{n}_T = \cos\theta_T \hat{\boldsymbol{x}} + \sin\theta_T \hat{\boldsymbol{y}} \quad , \quad \hat{\boldsymbol{n}}_R = \cos\theta_R \hat{\boldsymbol{x}} + \sin\theta_R \hat{\boldsymbol{y}} \quad .$$

and prove from the boundary conditions that $\theta_T = \theta_R = 0$*.]*
He means

$$\tilde{\boldsymbol{E}}_R = \hat{\boldsymbol{x}} E_{0R} e^{i(-k_1 z - \omega t)} \quad \tilde{\boldsymbol{B}}_R = -\hat{\boldsymbol{y}} \sqrt{\mu_1 \varepsilon_1} E_{0R} e^{i(-k_1 z - \omega t)}$$
$$\tilde{\boldsymbol{E}}_T = \hat{\boldsymbol{x}} E_{0T} e^{i(k_2 z - \omega t)} \quad \tilde{\boldsymbol{B}}_T = \hat{\boldsymbol{y}} \sqrt{\mu_2 \varepsilon_2} E_{0T} e^{i(k_2 z - \omega t)}$$

Polarization of transmitted and reflected light (continued)

for which we now have to write

$$\tilde{\boldsymbol{E}}_R = \hat{\boldsymbol{n}}_R E_{0R} e^{i(-k_1 z - \omega t)} \quad \tilde{\boldsymbol{B}}_R = -\left(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}_R\right) \sqrt{\mu_1 \varepsilon_1} E_{0R} e^{i(-k_1 z - \omega t)}$$
$$\tilde{\boldsymbol{E}}_T = \hat{\boldsymbol{n}}_T E_{0T} e^{i(k_2 z - \omega t)} \quad \tilde{\boldsymbol{B}}_T = \left(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}_T\right) \sqrt{\mu_2 \varepsilon_2} E_{0T} e^{i(k_2 z - \omega t)}$$

though $\tilde{\boldsymbol{E}}_I = \hat{\boldsymbol{x}} E_{0I} e^{i(k_1 z - \omega t)}$ and $\tilde{\boldsymbol{B}}_I = \hat{\boldsymbol{y}} \sqrt{\mu_1 \varepsilon_1} E_{0I} e^{i(k_1 z - \omega t)}$ still. The boundary conditions now give us

$$\tilde{E}_{0I} \hat{\boldsymbol{x}} + \tilde{E}_{0R} \hat{\boldsymbol{n}}_{\boldsymbol{R}} = \tilde{E}_{0T} \hat{\boldsymbol{n}}_{\boldsymbol{T}} \quad ,$$
$$\tilde{E}_{0I} \hat{\boldsymbol{y}} - \tilde{E}_{0R} \left(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}_{\boldsymbol{R}}\right) = \beta \tilde{E}_{0T} \left(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}_T\right) \quad .$$

In Cartesian components, the first of these is

$$\tilde{E}_{0I} + \tilde{E}_{0R} \cos\theta_R = \tilde{E}_{0T} \cos\theta_T \quad , \qquad x$$
$$\tilde{E}_{0R} \sin\theta_R = \tilde{E}_{0T} \sin\theta_T \quad . \qquad y$$

Polarization of transmitted and reflected light (continued)

The Cartesian components of the second one are

$$-\tilde{E}_{0R} \sin\theta_R = \beta \tilde{E}_{0T} \sin\theta_T \quad , \qquad x$$
$$\tilde{E}_{0I} - \tilde{E}_{0R} \cos\theta_R = \beta \tilde{E}_{0T} \cos\theta_T \quad . \qquad y$$

Compare the first y component to the second x component:

$$\tilde{E}_{0R} \sin\theta_R = \tilde{E}_{0T} \sin\theta_T \quad ,$$
$$-\tilde{E}_{0R} \sin\theta_R = \beta \tilde{E}_{0T} \sin\theta_T \quad \text{. Just add them:}$$
$$\tilde{E}_{0T} \sin\theta_T \left(1 + \beta\right) = 0 \quad \Rightarrow \quad \sin\theta_T = 0.$$
$$\Rightarrow \tilde{E}_{0R} \sin\theta_R = 0 \quad \Rightarrow \quad \sin\theta_R = 0 \quad .$$

Thus $\theta_T = \theta_R = 0$; that is, the incident, reflected and transmitted light have the same polarization.