Today in Physics 218: impedance of the vacuum, and Snell's Law

- The impedance of linear media
- Spacecloth
- Reflection and transmission of electromagnetic plane waves at interfaces: Snell's Law and the first Fresnel relation

Lots of small pieces of Eccosorb spacecloth (Emerson & Cuming Microwave Products Corp.)

Electromagnetic impedance of linear media

Enclose a hypothetical source of electromagnetic radiation with a surface. The power flowing through the surface will be

$$P(t) = \oint \boldsymbol{S} \cdot d\boldsymbol{a} = \frac{c}{4\pi} \oint (\boldsymbol{E} \times \boldsymbol{H}) \cdot d\boldsymbol{a} = \frac{c}{4\pi\mu} \oint \left(\boldsymbol{E} \times \left(\sqrt{\mu\varepsilon}\,\hat{\boldsymbol{k}} \times \boldsymbol{E} \right) \right) \cdot d\boldsymbol{a}$$

$$= \frac{c}{4\pi} \sqrt{\frac{\varepsilon}{\mu}} \oint \left(\hat{\boldsymbol{k}} \boldsymbol{E} \cdot \boldsymbol{E} - \boldsymbol{E} \left(\hat{\boldsymbol{k}} \cdot \boldsymbol{E} \right) \right) \cdot d\boldsymbol{a} = \frac{c}{4\pi} \sqrt{\frac{\varepsilon}{\mu}} \oint E^2 da_\perp$$

$$= \frac{c}{4\pi} \sqrt{\frac{\varepsilon}{\mu}} \oint E^2 d\ell_1 d\ell_2 = \frac{V_{\mathrm{eff}}^2}{Z} \quad ,$$

where $\boxed{Z = \frac{4\pi}{c} \sqrt{\frac{\mu}{\varepsilon}}}$ $\left[= \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_r \mu_0}{\varepsilon_r \varepsilon_0}} \text{ in MKS units} \right]$

Electromagnetic impedance of linear media (continued)

In vacuum,

$$Z = Z_0 = \frac{4\pi}{c} = 4.192 \times 10^{-10} \text{ sec cm}^{-1} \text{ in cgs units,}$$

$$= \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.73\ \Omega \text{ in MKS units.}$$

(Officially, ohms per *square*, as we'll see shortly.)

As in the case of the string, the amplitude ratios for light reflected and transmitted by the interface between two linear media can be expressed compactly in terms of impedance, because

$$\beta = \sqrt{\frac{\varepsilon_2 \mu_1}{\varepsilon_1 \mu_2}} = \frac{Z_1}{Z_2} \quad :$$

Electromagnetic impedance of linear media (continued)

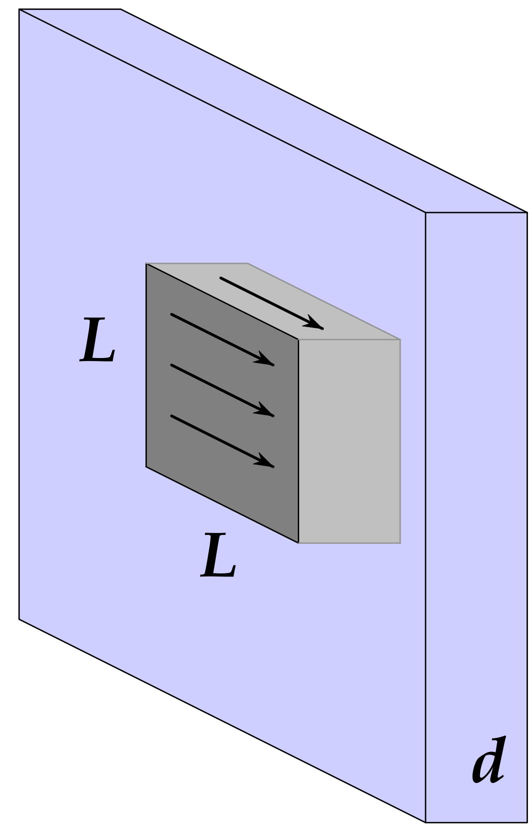
$$\tilde{E}_{0T} = \frac{2}{1+\beta}\tilde{E}_{0I} = \frac{2Z_2}{Z_2+Z_1}\tilde{E}_{0I} \quad ,$$

$$\tilde{E}_{0R} = \frac{1-\beta}{1+\beta}\tilde{E}_{0I} = \frac{Z_2-Z_1}{Z_2+Z_1}\tilde{E}_{0I} \quad .$$

- Note that the amplitude ratios come out simple in terms of impedance, even if $\mu \neq 1$.
- Note also that there's no reflected light if the impedances of the two media are equal (impedances **matched**). This will be true if $\varepsilon_2\mu_1 = \varepsilon_1\mu_2$; the media can be quite different and still have the same impedance.

Spacecloth

What does it mean for vacuum to have an impedance of 377Ω? (Most resistors have terminals!) Well, consider an infinite slab of material with resistivity ρ and thickness d, and consider further a square of side L within the slab.



For voltage difference V between two sides of the square, a current will encounter a resistance

$$R = \frac{\rho L}{A} = \frac{\rho L}{Ld} = \frac{\rho}{d} \quad .$$

This is true for any square, any size, any orientation.

Spacecloth (continued)

Space is like this: any square (any size) taken from a slab of vacuum exhibits an impedance for electric fields applied across the square.

- One may think of this impedance as originating in the induction of ***E*** and ***B*** from each other: try to change ***E***, for instance, and energy has to be put into ***B*** as well.
- A medium with resistivity and thickness such that its resistance per square is the same as vacuum would absorb light without reflection; that is, it would look to incident light as if it were an infinite vacuum.
- Such material exists: microwave engineers call it **spacecloth**.

Spacecloth (continued)

One prominent example of the use of highly optimized, durable spacecloth. (USAF photo.)

Reflection and transmission for oblique incidence

Now let's consider plane electromagnetic waves incident obliquely on an infinite, planar surface between two different linear media, a case which has no counterpart in the realm of strings. The geometry is shown on the next page, and the fields are

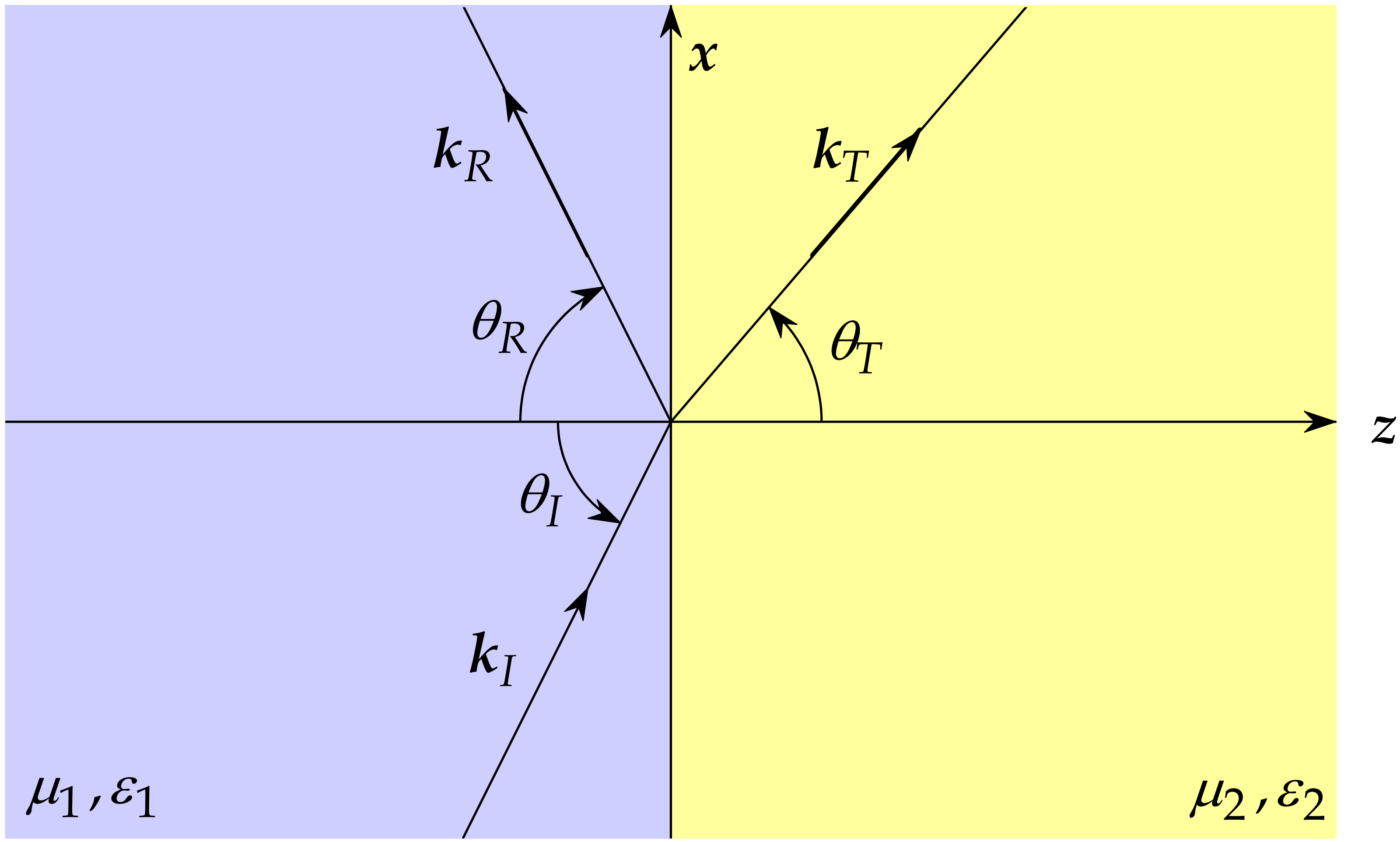
$$\tilde{\boldsymbol{E}}_I = \tilde{\boldsymbol{E}}_{0I} e^{i(\boldsymbol{k}_I\cdot\boldsymbol{r}-\omega t)} \qquad \tilde{\boldsymbol{B}}_I = \sqrt{\mu_1\varepsilon_1}\,\hat{\boldsymbol{k}}_I \times \tilde{\boldsymbol{E}}_I$$

$$\tilde{\boldsymbol{E}}_R = \tilde{\boldsymbol{E}}_{0R} e^{i(\boldsymbol{k}_R\cdot\boldsymbol{r}-\omega t)} \qquad \tilde{\boldsymbol{B}}_R = \sqrt{\mu_1\varepsilon_1}\,\hat{\boldsymbol{k}}_R \times \tilde{\boldsymbol{E}}_R$$

$$\tilde{\boldsymbol{E}}_T = \tilde{\boldsymbol{E}}_{0T} e^{i(\boldsymbol{k}_T\cdot\boldsymbol{r}-\omega t)} \qquad \tilde{\boldsymbol{B}}_T = \sqrt{\mu_2\varepsilon_2}\,\hat{\boldsymbol{k}}_T \times \tilde{\boldsymbol{E}}_T$$

and our problem is to find $\tilde{\boldsymbol{E}}_R, \tilde{\boldsymbol{E}}_T, \boldsymbol{k}_R$, and $\boldsymbol{k}_T$ in terms of $\tilde{\boldsymbol{E}}_I$ and $\boldsymbol{k}_I$. The former two will work out from the boundary conditions; the latter just from the fact that we are applying boundary conditions.

Reflection and transmission for oblique incidence (continued)



The $\boldsymbol{k}$s don't necessarily lie in a plane: we have to prove this.

Reflection and transmission for oblique incidence (continued)

Solution: At z = 0, write the continuity conditions: all four this time:

$$\varepsilon_1 E_{\perp,1} - \varepsilon_2 E_{\perp,2} = 0 \qquad B_{\perp,1} - B_{\perp,2} = 0$$

$$E_{\parallel,1} - E_{\parallel,2} = 0 \qquad \frac{1}{\mu_1} B_{\parallel,1} - \frac{1}{\mu_2} B_{\parallel,2} = 0$$

or,

$$\varepsilon_1 \left(\tilde{E}_{0Iz} + \tilde{E}_{0Rz} \right) = \varepsilon_2 \tilde{E}_{0Tz} \qquad \tilde{B}_{0Iz} + \tilde{B}_{0Rz} = \tilde{B}_{0Tz}$$

$$\tilde{E}_{0Ix} + \tilde{E}_{0Rx} = \tilde{E}_{0Tx} \qquad \frac{1}{\mu_1}\left(\tilde{B}_{0Ix} + \tilde{B}_{0Rx} \right) = \frac{1}{\mu_2} \tilde{B}_{0Tx}$$

$$\tilde{E}_{0Iy} + \tilde{E}_{0Ry} = \tilde{E}_{0Ty} \qquad \frac{1}{\mu_1}\left(\tilde{B}_{0Iy} + \tilde{B}_{0Ry} \right) = \frac{1}{\mu_2} \tilde{B}_{0Ty}$$

All of these are of the form

$$A e^{i\left(\boldsymbol{k}_I \cdot \boldsymbol{r} - \omega t\right)} + B e^{i\left(\boldsymbol{k}_R \cdot \boldsymbol{r} - \omega t\right)} = C e^{i\left(\boldsymbol{k}_T \cdot \boldsymbol{r} - \omega t\right)} \quad .$$

Reflection and transmission for oblique incidence (continued)

- As we've seen, this sort of result is not special to electromagnetic waves, but appears in boundary conditions for any sort of wave. Now, it is an interesting fact that, for nonzero constants *A, B, C, a, b,* and *c*, if

$$Ae^{iau} + Be^{ibu} = Ce^{icu} \quad ,$$

then $\boxed{a = b = c \text{ and } A + B = C,}$ for all u, as you will show in this week's homework (problem 9.15).

- For our boundary conditions, this means

$$\mathbf{k}_I \cdot \mathbf{r} = \mathbf{k}_R \cdot \mathbf{r} = \mathbf{k}_T \cdot \mathbf{r}$$

at $z = 0$, or

$$k_{\boldsymbol{Ix}} x + k_{Iy} y = k_{\boldsymbol{Rx}} x + k_{Ry} y = k_{\boldsymbol{Tx}} x + k_{Ty} y \quad .$$

Reflection and transmission for oblique incidence (continued)

- Furthermore, along the x axis, $k_{\boldsymbol{Ix}}x = k_{\boldsymbol{Rx}}x = k_{\boldsymbol{Tx}}x$, and similarly $k_{Iy}y = k_{Ry}y = k_{Ty}y$ along the y axis.
- If we were to orient the axes such that $k_{Iy} = 0$, then $k_{Ry} = k_{Ty} = 0$ as well, and the situation will look exactly like the picture a couple of slides ago, with all the $\boldsymbol{k}$s lying in a plane.
- Thus there is always such a plane:

$$\boxed{\boldsymbol{k}_I, \boldsymbol{k}_R, \text{ and } \boldsymbol{k}_T \text{ lie in a plane.}}$$

This plane is called the **plane of incidence.**

Reflection and transmission for oblique incidence (continued)

Now back to the figure. From $k_{\boldsymbol{Ix}}x = k_{\boldsymbol{Rx}}x = k_{\boldsymbol{Tx}}x$ at $z = 0$ we also get

$$k_{\boldsymbol{I}} \sin\theta_I = k_{\boldsymbol{R}} \sin\theta_R = k_{\boldsymbol{T}} \sin\theta_T \quad .$$

But $\boldsymbol{k_I} = \boldsymbol{k_R} = \omega/v_1 = \omega n_1/c$, so the first equation gives us the familiar mirror reflection law:

$$\sin\theta_I = \sin\theta_R \quad \Rightarrow \quad \boxed{\theta_I = \theta_R} \quad .$$

And $\boldsymbol{k_T} = \omega/v_2$, so

$$\boxed{\frac{\sin\theta_T}{\sin\theta_I} = \frac{k_{\boldsymbol{I}}}{k_{\boldsymbol{T}}} = \frac{v_2}{v_1} = \frac{n_1}{n_2} \quad .} \qquad \text{Snell's Law}$$

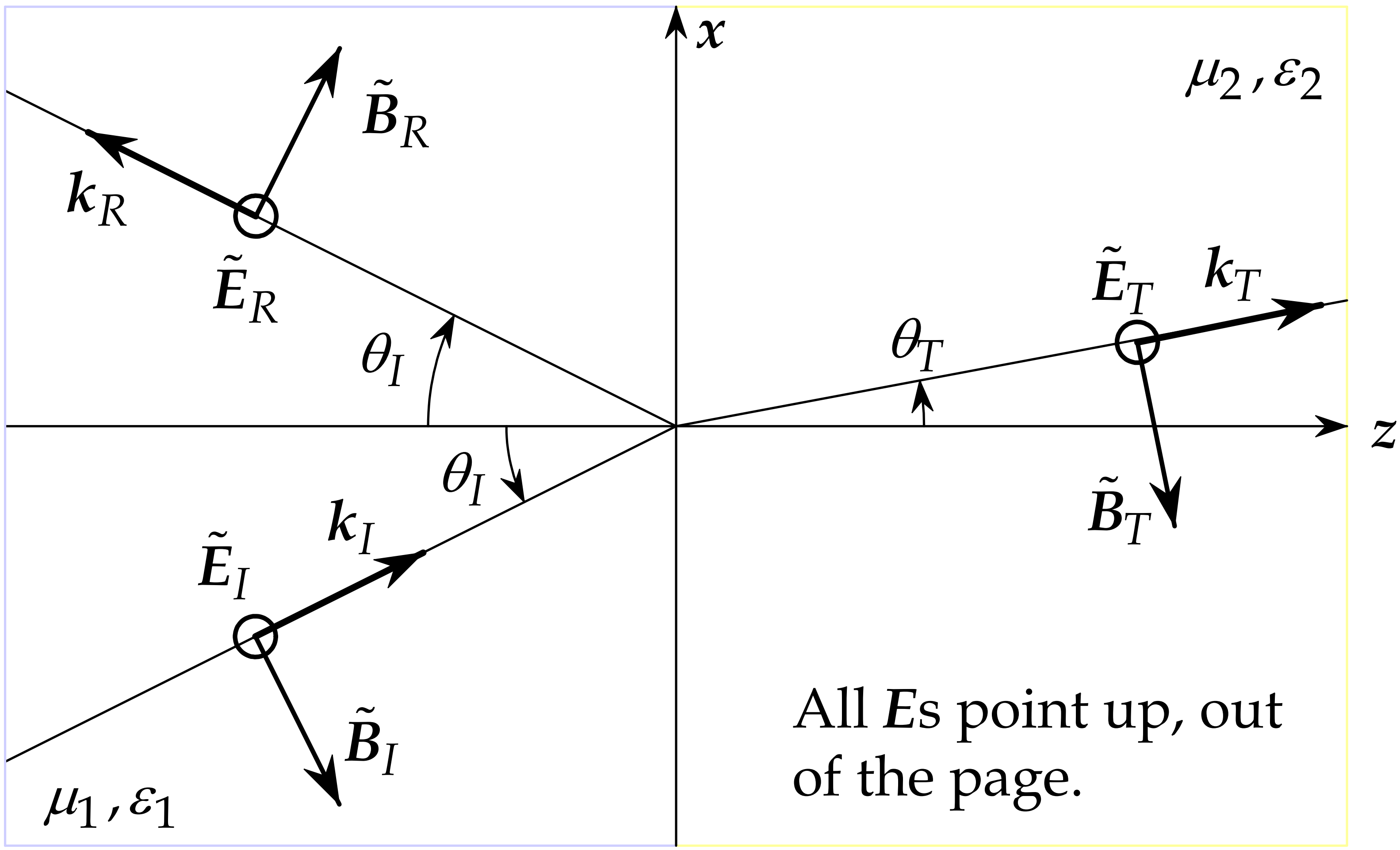
Since the conditions arise just from continuity, rather than the physics that gives rise in this case to continuity, these laws will apply to waves quite generally.

Reflection and transmission for oblique incidence (continued)

Since the wave vectors all lie in the incidence plane and the fields are transverse, we can treat separately the cases of light with $\boldsymbol{E}$ polarized perpendicular to, or parallel to, the plane of incidence.

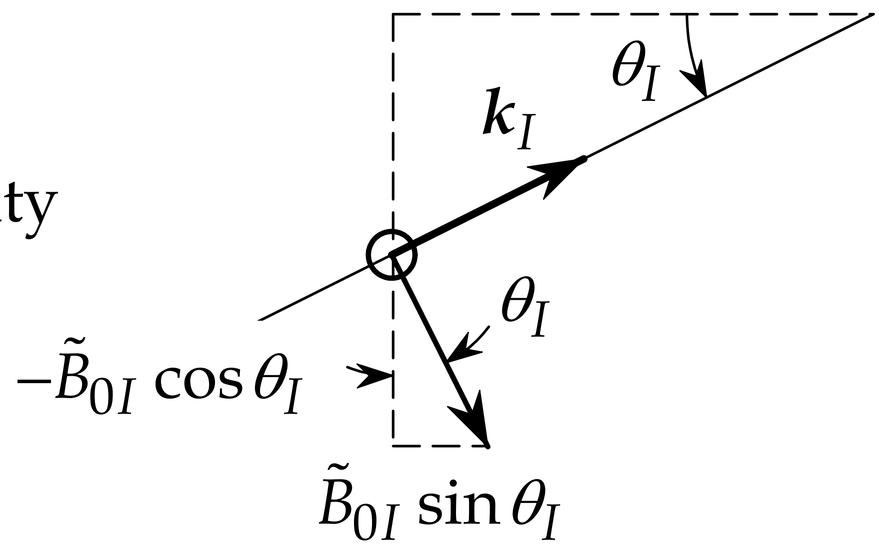
First we'll treat the perpendicular case, also known as problem !9.16.

E perpendicular to incidence plane



E perpendicular to incidence plane (continued)

Let's use the same two boundary conditions that we used for normal incidence, continuity of the parallel components. (We only need two, because $\tilde{E}_{0R}$ and $\tilde{E}_{0T}$ are the only unknowns left.) At $z = 0$, these are



$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \quad ,$$

$$\frac{1}{\mu_1}\left(-\sqrt{\mu_1\varepsilon_1}\,\tilde{E}_{0I}\cos\theta_I + \sqrt{\mu_1\varepsilon_1}\,\tilde{E}_{0R}\cos\theta_I\right) = -\frac{1}{\mu_2}\sqrt{\mu_2\varepsilon_2}\,\tilde{E}_{0T}\cos\theta_T .$$

The $D_\perp$ condition tells us nothing this time; the $B_\perp$ one tells us nothing new:

E perpendicular to incidence plane (continued)

$$\sqrt{\mu_1\varepsilon_1}\,\tilde{E}_{0I}\sin\theta_I + \sqrt{\mu_1\varepsilon_1}\,\tilde{E}_{0R}\sin\theta_I = \sqrt{\mu_2\varepsilon_2}\,\tilde{E}_{0T}\sin\theta_T \quad .$$

Using Snell's Law on this,

$$n_1\sin\theta_I = n_2\sin\theta_T \quad \Rightarrow \quad \sqrt{\mu_1\varepsilon_1}\sin\theta_I = \sqrt{\mu_2\varepsilon_2}\sin\theta_T \quad ,$$

we get

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \quad ,$$

which we already knew.

- We will generally find this to be true in our upcoming work: one boundary condition will tell us nothing, and two of the remaining ones will be equivalent.
- Also generally, though, the parallel-component BCs tell us different things, so well use them a lot.

E perpendicular to incidence plane (continued)

Now define $\alpha = \frac{\cos\theta_T}{\cos\theta_I}$, $\beta = \sqrt{\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}} = \frac{Z_1}{Z_2}$,

and divide the $H_{||}$ equation by $\sqrt{\mu_1\varepsilon_1}\cos\theta_I$:

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \quad ,$$

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \alpha\beta\tilde{E}_{0T} \quad ,$$

almost like the string again. Add them to get

$$2\tilde{E}_{0I} = (1+\alpha\beta)\tilde{E}_{0T} \quad \Rightarrow \quad \tilde{E}_{0T} = \frac{2\tilde{E}_{0I}}{1+\alpha\beta} \quad .$$

Same as before, if $\alpha = 1$.

Multiply the first one by $\alpha\beta$ and subtract to get

$$(\alpha\beta - 1)\tilde{E}_{0I} + (\alpha\beta + 1)\tilde{E}_{0R} = 0 \quad \Rightarrow \quad \tilde{E}_{0R} = \frac{1-\alpha\beta}{1+\alpha\beta}\tilde{E}_{0I} \quad .$$