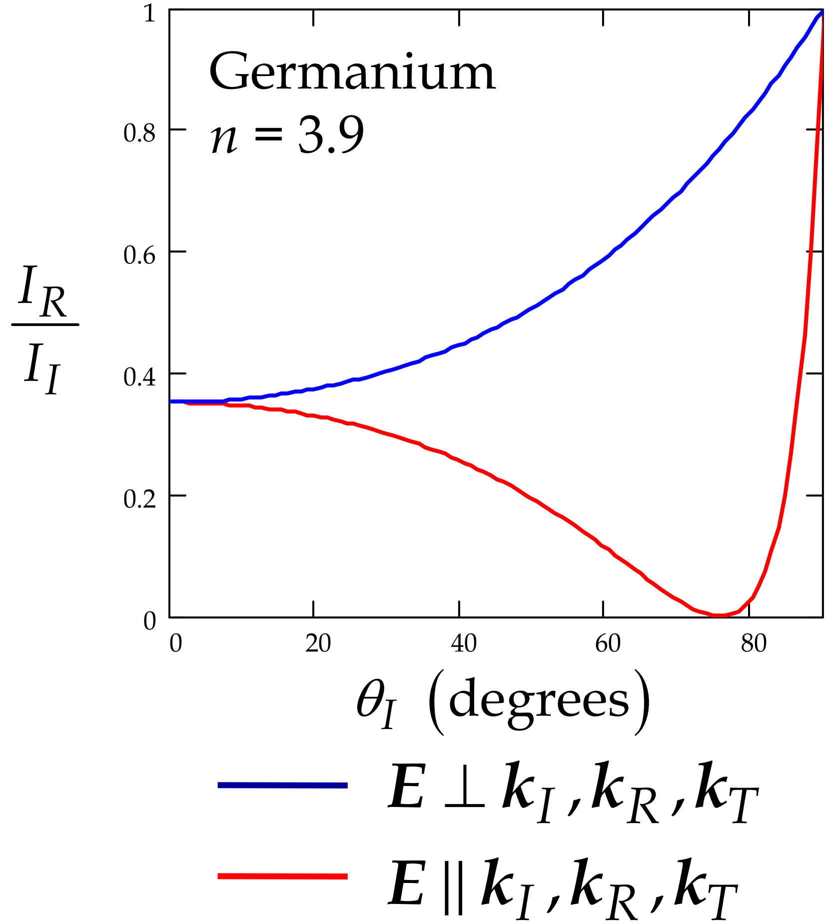
Today in Physics 218: Fresnel's equations

- Transmission and reflection with ***E*** parallel to the incidence plane
- The Fresnel equations
- Total internal reflection
- Polarization on reflection
- Interference



Last time: *E* perpendicular to the incidence plane

…for which we obtained

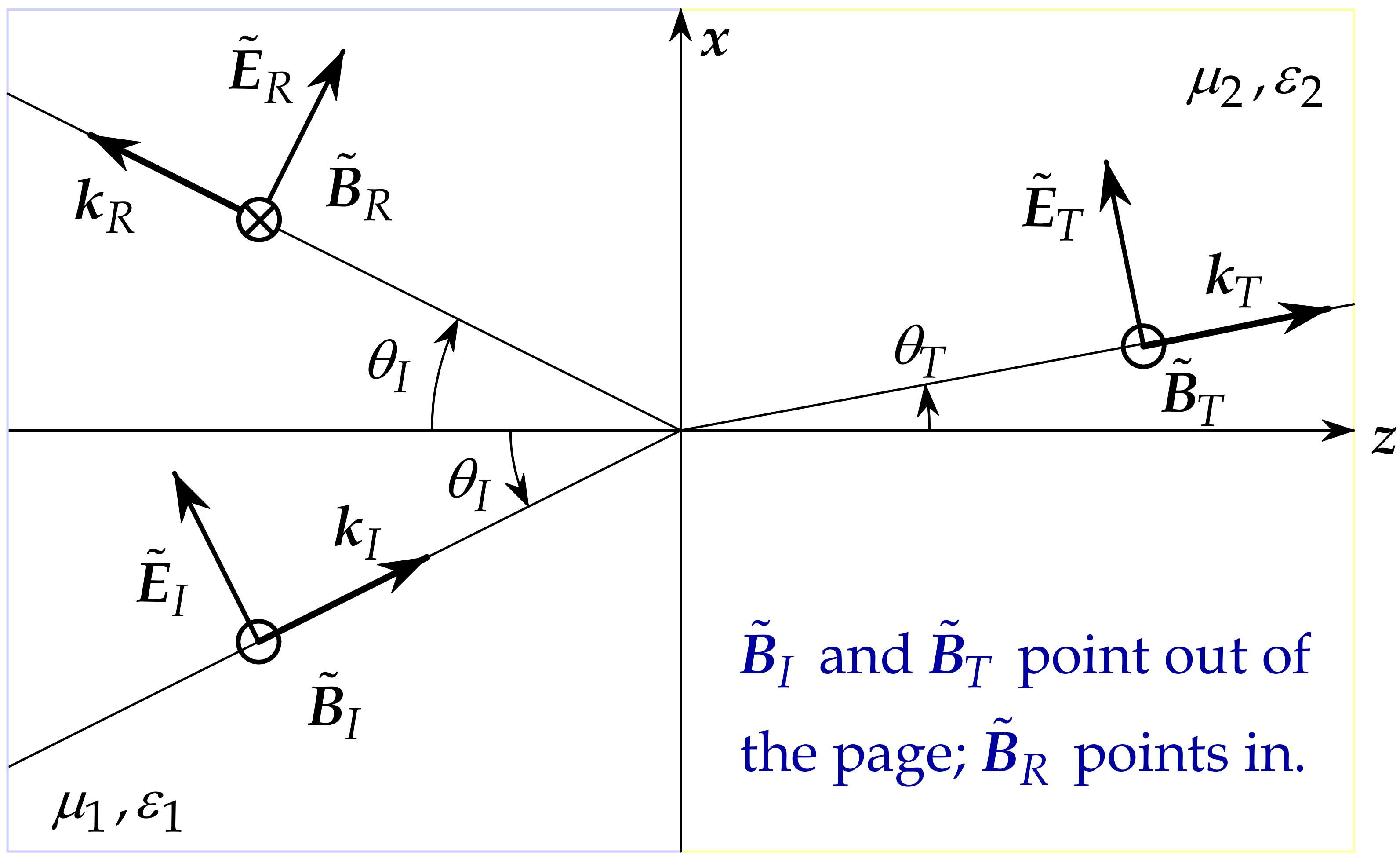
$$\tilde{E}_{0T} = \frac{2\tilde{E}_{0I}}{1+\alpha\beta} \quad \text{and} \quad \tilde{E}_{0R} = \frac{1-\alpha\beta}{1+\alpha\beta}\tilde{E}_{0I}$$

for the transmitted and reflected amplitudes of ***E***, where

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I} \quad \text{, and} \quad \beta = \sqrt{\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}} = \frac{Z_1}{Z_2} \quad .$$

Now, to complete the picture, we need to consider incident light with ***E*** polarized in the plane of incidence.

E parallel to the incidence plane



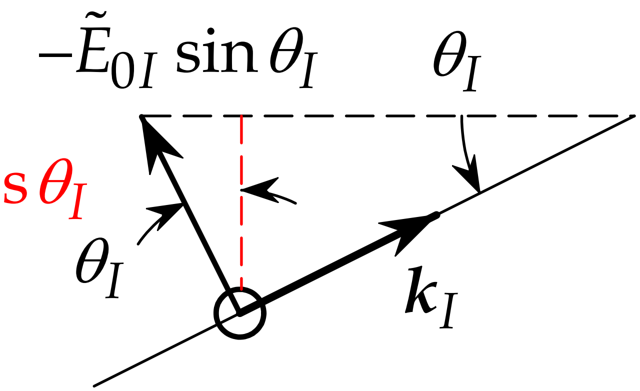
$\boldsymbol{E}$ parallel to the incidence plane (continued)

Again we will use the boundary conditions on $\boldsymbol{E}_{\|}$ and $\boldsymbol{H}_{\|}$:

$$\tilde{E}_{0I}\cos\theta_I + \tilde{E}_{0R}\cos\theta_I = \tilde{E}_{0T}\cos\theta_T \quad ,$$

$$\frac{1}{\mu_1}\left(\sqrt{\mu_1\varepsilon_1}\,\tilde{E}_{0I} - \sqrt{\mu_1\varepsilon_1}\,\tilde{E}_{0R}\right) = \frac{1}{\mu_2}\sqrt{\mu_2\varepsilon_2}\,\tilde{E}_{0T} \quad .$$

This time it's the $\boldsymbol{B}_{\perp}$ boundary condition that tell us nothing besides 0 = 0, and the $\boldsymbol{D}_{\perp}$ boundary condition that turns out to be identical to the $\boldsymbol{E}_{\|}$ boundary condition.



E parallel to the incidence plane (continued)

Now divide the first of these equations by $\cos\theta_I$, and divide the other one by $\sqrt{\varepsilon_1/\mu_1}$:

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \frac{\cos\theta_T}{\cos\theta_I}\tilde{E}_{0T} = \alpha\tilde{E}_{0T} \quad ,$$

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \sqrt{\frac{\mu_1\varepsilon_2}{\mu_2\varepsilon_1}}\tilde{E}_{0T} = \beta\tilde{E}_{0T} \quad ,$$

where, again,

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I} \quad \text{and} \quad \beta = \sqrt{\frac{\mu_1\varepsilon_2}{\mu_2\varepsilon_1}} = \frac{Z_1}{Z_2} \quad .$$

It's a simple matter to solve these equations for the reflected and transmitted field amplitudes; in fact it's the same as we already did three times. The result is:

E parallel to the incidence plane (continued)

$$\tilde{E}_{0T} = \frac{2\tilde{E}_{0I}}{\alpha+\beta} \quad \text{and} \quad \tilde{E}_{0R} = \frac{\alpha-\beta}{\alpha+\beta}\tilde{E}_{0I} \quad .$$

All of these results comprise the **Fresnel equations**:

$$\boldsymbol{E} \perp \boldsymbol{k}_I, \boldsymbol{k}_R, \boldsymbol{k}_T : \quad \tilde{E}_{0T} = \frac{2\tilde{E}_{0I}}{1+\alpha\beta} \quad , \quad \tilde{E}_{0R} = \frac{1-\alpha\beta}{1+\alpha\beta}\tilde{E}_{0I} \quad .$$

$$\boldsymbol{E} \parallel \boldsymbol{k}_I, \boldsymbol{k}_R, \boldsymbol{k}_T : \quad \tilde{E}_{0T} = \frac{2\tilde{E}_{0I}}{\alpha+\beta} \quad , \quad \tilde{E}_{0R} = \frac{\alpha-\beta}{\alpha+\beta}\tilde{E}_{0I} \quad .$$

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I} \cong \frac{1}{\cos\theta_I}\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_I\right)^2}$$

$$\beta = \sqrt{\mu_1\varepsilon_2/\mu_2\varepsilon_1} = Z_1/Z_2 \cong \sqrt{\varepsilon_2/\varepsilon_1} = n_2/n_1$$

The $\cong$ applies for almost all non-conductors.

Normal incidence

We shall now turn to some interesting and useful implications of Fresnel's equations, and other relations that can be obtained in the same manner.

- **Normal incidence**. Both pairs of Fresnel equations reduce to the same expression we derived before for incidence angle zero $\left(\alpha = \cos\theta_T / \cos\theta_I = 1\right)$:

$$\tilde{E}_{0T} = \frac{2\tilde{E}_{0I}}{1+\beta} \quad \text{and} \quad \tilde{E}_{0R} = \frac{1-\beta}{1+\beta}\tilde{E}_{0I} \quad ,$$

as they must if they're correct.

Total reflection

- Suppose $\mu_1\varepsilon_1 > \mu_2\varepsilon_2 \ (n_1 > n_2)$. Then, at θ_I given by

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_{IC} = 1 \quad ,$$

the transmitted wave is parallel to the interface, and for values of $\theta_I > \theta_{IC}$, we get $\sin\theta_T > 1$!
What does this mean? In this condition,

$$\cos\theta_T = \sqrt{1-\sin^2\theta_T} = \sqrt{\boxed{1-\frac{n_1^2}{n_2^2}\sin^2\theta_I}} \longrightarrow \boxed{<0\,!}$$

$$= i\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_I - 1} \quad .$$

Total reflection (continued)

- Now the $\sqrt{}$ is real, but the projection of the wavevector is imaginary:

$$\tilde{\boldsymbol{E}}_T = \tilde{\boldsymbol{E}}_{0T} e^{i(\boldsymbol{k}_T \cdot \boldsymbol{r} - \omega t)} = \tilde{\boldsymbol{E}}_{0T} e^{i(k_T z \cos\theta_I + k_T x \sin\theta_I - \omega t)}$$

$$= \tilde{\boldsymbol{E}}_{0T} e^{\boxed{-k_T z \sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_I - 1}}} e^{i(k_T x \sin\theta_I - \omega t)} \quad .$$

Real!

The transmitted wave amplitude decreases exponentially with increasing z (exponential *attenuation*): no energy is transmitted to large distances, all energy is reflected, at incidence angles greater than θ_{IC}.

Total reflection (continued)

For example: flint glass $(n_1 = 1.5)$ in air $(n_2 = 1)$ has

$$\sin\theta_{IC} = \frac{n_2}{n_1} = \frac{1}{1.5} \quad \Rightarrow \quad \theta_{IC} = 41.8^\circ \quad .$$

At $\theta_I > \theta_{IC}$,light is totally reflected. This is how light fibers work.

(Now you know how much a light fiber can be bent before it stops working.)

Polarization on reflection

- If $\tilde{E}_{0I}$ is parallel to the plane of incidence, then

$$\tilde{E}_{0R} = \frac{\alpha - \beta}{\alpha + \beta}\tilde{E}_{0I} = 0 \quad \text{at} \quad \alpha = \beta \quad ;$$

that is, all light polarized in the plane of incidence must be transmitted (none reflected) if this is true. The incidence angle that corresponds to this is obtained from:

$$\alpha = \frac{\cos\theta_{TB}}{\cos\theta_{IB}} = \frac{1}{\cos\theta_{IB}}\sqrt{1 - \left(\frac{n_1}{n_2}\sin\theta_{IB}\right)^2} = \beta$$

$$1 - \left(\frac{n_1}{n_2}\sin\theta_{IB}\right)^2 = \beta^2\left(1 - \sin^2\theta_{IB}\right)$$

Polarization on reflection (continued)

$$1-\beta^2 = \left[\left(\frac{n_1}{n_2}\right)^2 - \beta^2\right]\sin^2\theta_{IB} \quad \Rightarrow \quad \sin\theta_{IB} = \sqrt{\frac{1-\beta^2}{\left(\frac{n_1}{n_2}\right)^2 - \beta^2}} \quad .$$

Typically, $\mu_1 = \mu_2 = 1$, so $\beta = n_2/n_1$:

$$\sin\theta_{IB} = \sqrt{\frac{1-\beta^2}{\left(\frac{1}{\beta}\right)^2 - \beta^2}} = \sqrt{\beta^2 \frac{1-\beta^2}{\left(1-\beta^2\right)\left(1+\beta^2\right)}} = \sqrt{\frac{\beta^2}{\left(1+\beta^2\right)}} \quad .$$

But, $\sin\theta = \tan\theta\cos\theta = \sqrt{\tan^2\theta\cos^2\theta} = \sqrt{\frac{\tan^2\theta}{1+\tan^2\theta}} \quad .$

Compare these last two results:

Polarization on reflection (continued)

$$\boxed{\tan\theta_{IB} = \beta = \frac{n_2}{n_1} \quad .}$$

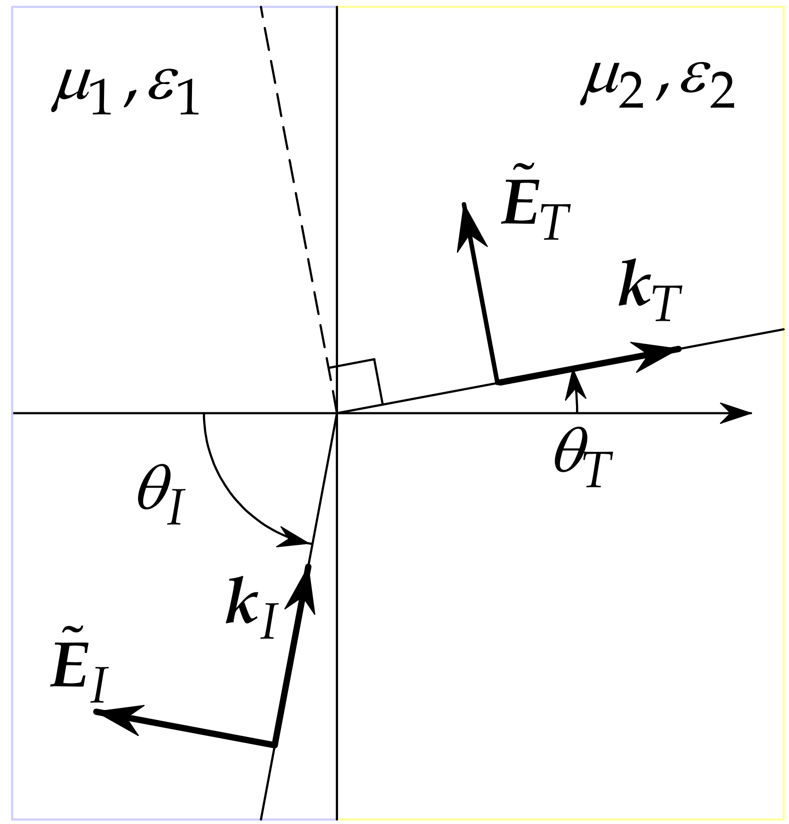
Brewster's angle

Note also that for $\theta_I = \theta_{IB}$,

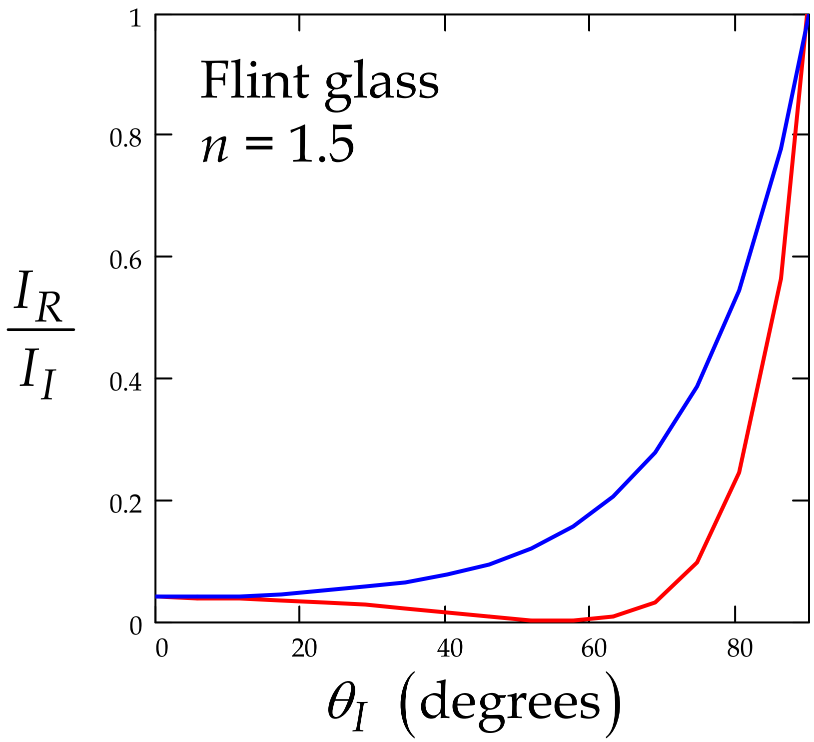
$$\frac{n_2}{n_1} = \tan\theta_I = \frac{\sin\theta_I}{\cos\theta_I} = \frac{\sin\theta_I}{\sin\left(\frac{\pi}{2} - \theta_I\right)}$$

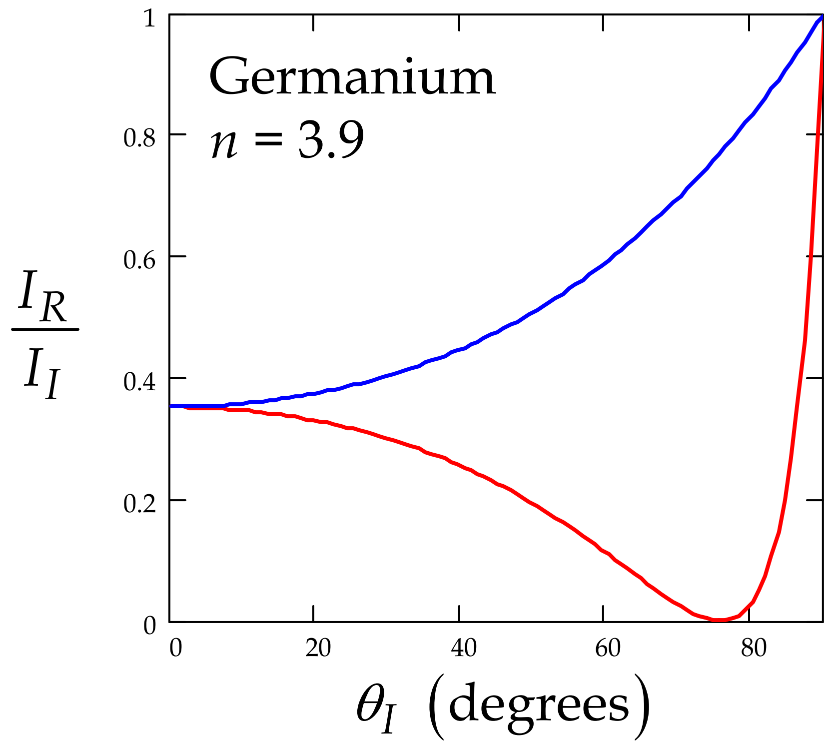
$$= \frac{\sin\theta_I}{\sin\theta_T} \quad \text{by Snell's law.}$$

Thus $\theta_I + \theta_T = \pi/2$ for light incident at Brewster's angle; the transmitted light travels perpendicular to the direction of reflection.



Polarization on reflection (continued)





Fraction of incident intensity reflected, for materials in air.

E perpendicular to plane of incidence

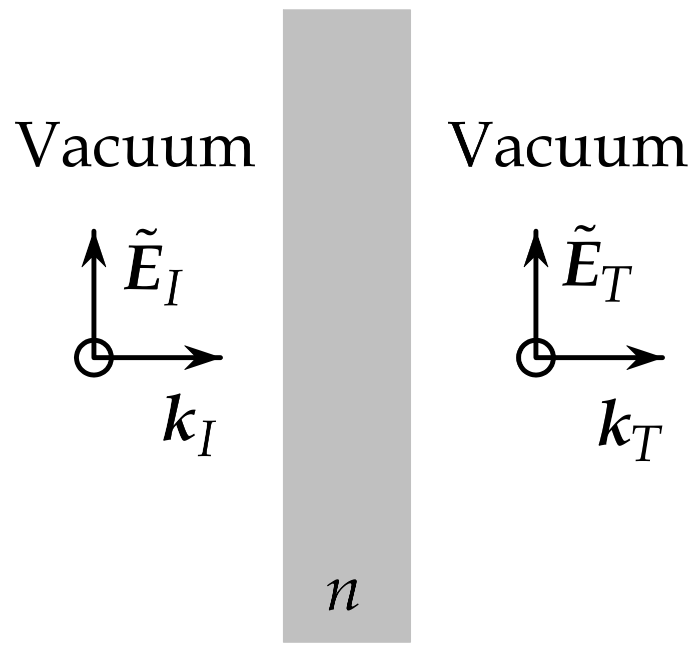
E parallel to plane of incidence

Interference

You will show, in problem set #3, that the intensity transmitted by a plane parallel slab of dielectric material with refractive index n and thickness d, at normal incidence, is

$$T = \frac{I_T}{I_I} = \frac{1}{1+\left(\dfrac{n^2-1}{2n}\sin nkd\right)^2} \quad ,$$

where k is the wavenumber in vacuum.



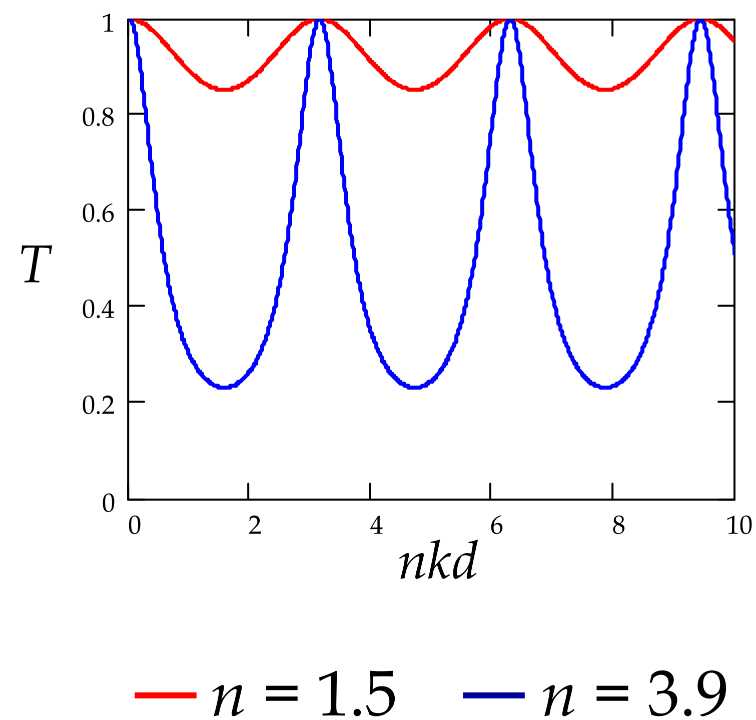
Interference (continued)

Look at this formula carefully and you'll see that $T = 1$ (100% transmission) for certain values of k,

$$T = 1 \quad \text{for} \quad nkd = m\pi, \quad m = 0, 1, 2, \ldots$$

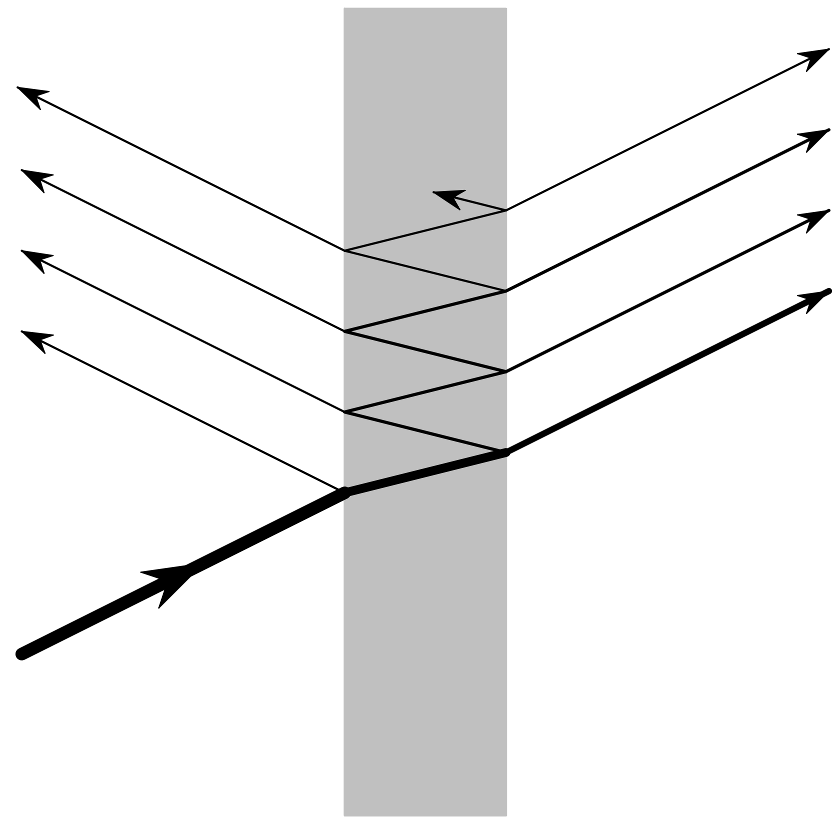
because the sine term vanishes there. The peaks get sharper with increasing refractive index n. What is the origin of the peaks?

(Interference, obviously, but how?)



Interference (continued)

One can think of the transmission as the result of a large number of internal reflections and transmissions. At **normal** incidence, all of the transmitted waves would be in phase (and interfere constructively) if an integer number of wavelengths was covered for each two internal reflections:



$$2d = m\frac{\lambda}{n} = m\frac{2\pi}{nk} \quad \Rightarrow \quad nkd = m\pi\,, m = 0,1,2,\ldots$$

Just like the formula!

Interference (continued)

- This, too, can be solved as a boundary-value problem, by considering the boundary conditions at both surfaces, and supposing there are transmitted and reflected waves propagating inside the slab, as well as the incident and reflected waves on the incidence side, and just a transmitted wave on the far side.
- The boundary conditions give us a system of four equations in four unknowns, in this case.
- It can certainly be done this way, and we'll see an example. I'll also introduce a much handier way, that can be used easily to find transmission and reflection through an arbitrary number of dielectric slabs and surfaces.