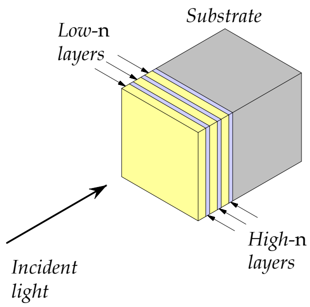
Today in Physics 218: stratified linear media I

- Interference in layers of linear media
- Transmission and reflection in stratified linear media, viewed as a boundary-value problem
- Matrix formulation of the fields at the interfaces

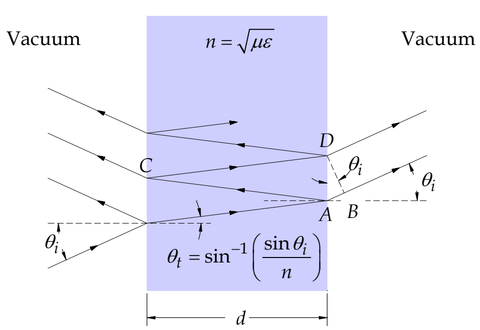


Interference in layers of linear media

As a preamble to the general question of transmission and reflection by stratified media, we will ask a simpler one: what is the condition for completely constructive interference in a single layer of linear material?

- Consider two plane-parallel, partially reflecting surfaces separated by a linear medium with refractive index $n = \sqrt{\mu\varepsilon}$ and thickness d (next slide).
- It doesn't matter what the index of refraction outside the reflectors is, but we will assume here that it is unity (vacuum) on both sides.
- If the transmitted or reflected rays are focussed then the waves interfere. By calculating the path-length differences, we can find out how they interfere.

Interference in layers of linear media (continued)



Interference in layers of linear media (continued)

- The path length difference between any two successive transmitted waves is the same. For the first set, that's the length between *AB* and *ACD:*

$$AB = 2d\tan\theta_t \sin\theta_i = 2dn\frac{\sin^2\theta_t}{\cos\theta_t} \quad ,$$

$$\left.\begin{array}{l} n\sin\theta_t = \sin\theta_i \\ AD = 2d\tan\theta_t \end{array}\right\} \Rightarrow \quad ACD = \frac{2d}{\cos\theta_t} \quad .$$

- The wavelength is λ in vacuum and λ/n in the medium between the reflectors, so

$$\delta(AB) = 2\pi\frac{AB}{\lambda} = \frac{4\pi dn}{\lambda\cos\theta_t}\sin^2\theta_t \quad ,$$

$$\delta(ACD) = 2\pi\frac{nACD}{\lambda} = \frac{4\pi dn}{\lambda\cos\theta_t} \quad .$$

Interference in layers of linear media (continued)

- If the phase difference is an integer multiple of 2π, then the interference between the two wavefronts corresponding to these paths is completely constructive:

$$\Delta\delta = \delta(ACD) - \delta(AB) = \frac{4\pi dn}{\lambda\cos\theta_t}\left(1-\sin^2\theta_t\right) = \frac{4\pi dn\cos\theta_t}{\lambda} \quad ,$$
$$= 2\pi m \qquad (m = 0,1,2,\ldots).$$

- Thus there are maxima in the spectrum of the transmission of the dielectric slab, at wavelengths given by

$$\lambda_m = \frac{2dn\cos\theta_t}{m} \qquad (m = 0,1,2,\ldots).$$

This, BTW, is the principle of the *Fabry-Perot interferometer*.

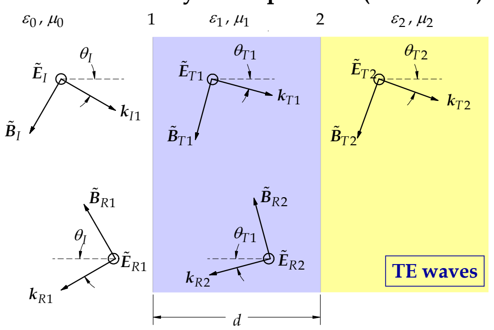
Transmission and reflection in stratified linear media, viewed as a boundary-value problem

Now we will set up the general solution to the problem of the transmission and reflection by a plane parallel layer, and find thereby a method for dealing with as many layers as we want.

Consider light propagating in one medium, incident obliquely on a layer of a second medium, and emerging into a third (next slide). What are the amplitudes of the transmitted and reflected waves?

- As before, this can be broken into two parts, one with light polarized perpendicular to the plane of incidence (**TE**), and one with ***E*** parallel to the plane of incidence (**TM**). We'll do TE first, and fill out the boundary conditions at the surfaces.

Transmission and reflection in stratified linear media as a boundary-value problem (continued)



Transmission and reflection in stratified linear media as a boundary-value problem (continued)

- The electric fields look generically like this:

  $\tilde{E} = \tilde{E}_0 e^{i(n\mathbf{k}\cdot\mathbf{r}-\omega t)}$ for waves propagating toward $+z$,

  $\tilde{E} = \tilde{E}_0 e^{i(-n\mathbf{k}\cdot\mathbf{r}-\omega t)}$ the other way.

  And of course $\tilde{B} = \sqrt{\mu\varepsilon}\,\hat{k}\times\tilde{E}$.

- At surface 1, the boundary conditions on $E_{\parallel}$ and $H_{\parallel}$ are

$$\tilde{E}_{\parallel,1} = \tilde{E}_{0I} + \tilde{E}_{0R1} = \tilde{E}_{0T1} + \tilde{E}_{0R2} \quad ,$$

$$\tilde{H}_{\parallel,1} = \frac{1}{\mu_0}\left(\tilde{B}_{0I}\cos\theta_I - \tilde{B}_{0R1}\cos\theta_I\right)$$

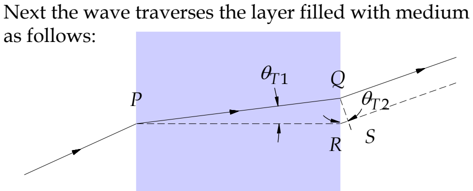
$$= \frac{1}{\mu_1}\left(\tilde{B}_{0T1}\cos\theta_{T1} - \tilde{B}_{0R2}\cos\theta_{T1}\right) \quad ,$$

Transmission and reflection in stratified linear media as a boundary-value problem (continued)

or $\tilde{E}_{0I} + \tilde{E}_{0R1} = \tilde{E}_{0T1} + \tilde{E}_{0R2} \quad ,$

$$\sqrt{\frac{\varepsilon_0}{\mu_0}}\cos\theta_I\left(\tilde{E}_{0I} - \tilde{E}_{0R1}\right) = \sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\theta_{T1}\left(\tilde{E}_{0T1} - \tilde{E}_{0R2}\right) \quad .$$

- Next the wave traverses the layer filled with medium #1, as follows:



Transmission and reflection in stratified linear media as a boundary-value problem (continued)

- As a wave crosses the slab it travels a distance

$$PQ = d/\cos\theta_{T1} \quad .$$

Compared to the undisplaced wave that would have resulted if the slab were not there, it undergoes a phase change of

$$\delta_1 = k_1\ell_1 + k_2\ell_2 = \frac{2\pi n_1}{\lambda}PQ - \frac{2\pi n_2}{\lambda}RS$$

$$= \frac{2\pi n_1 d}{\lambda\cos\theta_{T1}} - \frac{2\pi n_2}{\lambda}d\tan\theta_{T1}\sin\theta_{T2}$$

$$= \frac{2\pi n_1 d}{\lambda\cos\theta_{T1}}\left(1-\sin^2\theta_{T1}\right) = \frac{2\pi n_1 d}{\lambda}\cos\theta_{T1} \quad .$$

(half that of the two reflections in slide 5)

Transmission and reflection in stratified linear media as a boundary-value problem (continued)

- Thus the $\boldsymbol{E}_{\parallel}$ and $\boldsymbol{H}_{\parallel}$ boundary conditions at surface 2 are

$$\tilde{E}_{\parallel,2} = \tilde{E}_{0T1}e^{i\delta_1} + \tilde{E}_{0R2}e^{-i\delta_1} = \tilde{E}_{0T2}e^{i\delta_1} \quad ,$$

$$\tilde{H}_{\parallel,2} = \sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\theta_{T1}\left(\tilde{E}_{0T1}e^{i\delta_1} - \tilde{E}_{0R2}e^{-i\delta_1}\right) = \sqrt{\frac{\varepsilon_2}{\mu_2}}\cos\theta_{T2}\tilde{E}_{0T2}e^{i\delta_1} \quad .$$

- At this point we have four equations that we can solve for the four unknown amplitudes, $\tilde{E}_{0R1}$, $\tilde{E}_{0R2}$, $\tilde{E}_{0T1}$, and $\tilde{E}_{0T2}$, for the TE case. You can proceed directly in this manner, to solve a couple of the problems in this week's homework (e.g. Crawford 5.21, Griffiths !9.34). But it would be incredibly tedious to treat more than one layer like this. Fortunately there's a better way…

Matrix formulation of the fields at the interfaces

- The clever way to solve these problems starts by rearranging the boundary conditions to obtain relations between the fields at the two interfaces. $\tilde{E}_{0T1}$ and $\tilde{E}_{0R2}$ appear in both sets of boundary conditions, so solve the latest result for these two amplitudes:

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\theta_{T1}\tilde{E}_{\parallel,2} = \sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\theta_{T1}\left(\tilde{E}_{0T1}e^{i\delta_1} + \tilde{E}_{0R2}e^{-i\delta_1}\right)$$

$$+ \qquad \tilde{H}_{\parallel,2} = \sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\theta_{T1}\left(\tilde{E}_{0T1}e^{i\delta_1} - \tilde{E}_{0R2}e^{-i\delta_1}\right)$$

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\theta_{T1}\tilde{E}_{\parallel,2} + \tilde{H}_{\parallel,2} = 2\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\theta_{T1}\tilde{E}_{0T1}e^{i\delta_1} \quad \text{, or}$$

Matrix formulation of the fields at the interfaces (continued)

$$\tilde{E}_{0T1} = \frac{1}{2} e^{-i\delta_1} \left(\tilde{E}_{\|,2} + \sqrt{\frac{\mu_1}{\varepsilon_1}} \frac{\tilde{H}_{\|,2}}{\cos\theta_{T1}} \right) .$$

- Put this back in the surface-2 boundary conditions, and solve for $\tilde{E}_{0R2}$:

$$\tilde{E}_{\|,2} = \frac{1}{2} e^{-i\delta_1} \left(\tilde{E}_{\|,2} + \sqrt{\frac{\mu_1}{\varepsilon_1}} \frac{\tilde{H}_{\|,2}}{\cos\theta_{T1}} \right) e^{i\delta_1} + \tilde{E}_{0R2} e^{-i\delta_1} \quad \text{, or}$$

$$\tilde{E}_{0R2} = \frac{1}{2} e^{i\delta_1} \left(\tilde{E}_{\|,2} - \sqrt{\frac{\mu_1}{\varepsilon_1}} \frac{\tilde{H}_{\|,2}}{\cos\theta_{T1}} \right) .$$

- Now put both of these into the surface-1 boundary conditions:

Matrix formulation of the fields at the interfaces (continued)

$$\tilde{E}_{\|,1} = \frac{1}{2} e^{-i\delta_1} \left(\tilde{E}_{\|,2} + \sqrt{\frac{\mu_1}{\varepsilon_1}} \frac{\tilde{H}_{\|,2}}{\cos\theta_{T1}} \right) + \frac{1}{2} e^{i\delta_1} \left(\tilde{E}_{\|,2} - \sqrt{\frac{\mu_1}{\varepsilon_1}} \frac{\tilde{H}_{\|,2}}{\cos\theta_{T1}} \right)$$

$$= \tilde{E}_{\|,2} \cos\delta_1 - \tilde{H}_{\|,2} \sqrt{\frac{\mu_1}{\varepsilon_1}} \frac{i \sin\delta_1}{\cos\theta_{T1}} \quad ,$$

$$\tilde{H}_{\|,1} = \sqrt{\frac{\varepsilon_1}{\mu_1}} \cos\theta_{T1} \left[\frac{1}{2} e^{-i\delta_1} \left(\tilde{E}_{\|,2} + \sqrt{\frac{\mu_1}{\varepsilon_1}} \frac{\tilde{H}_{\|,2}}{\cos\theta_{T1}} \right) - \frac{1}{2} e^{i\delta_1} \left(\tilde{E}_{\|,2} - \sqrt{\frac{\mu_1}{\varepsilon_1}} \frac{\tilde{H}_{\|,2}}{\cos\theta_{T1}} \right) \right]$$

$$= -\tilde{E}_{\|,2} \sqrt{\frac{\varepsilon_1}{\mu_1}} \cos\theta_{T1} i \sin\delta_1 + \tilde{H}_{\|,2} \cos\delta_1 \quad .$$

Matrix formulation of the fields at the interfaces (continued)

- Now define

$$Y_{1,TE} = \sqrt{\frac{\varepsilon_1}{\mu_1}} \cos\theta_{T1} = \frac{4\pi}{c} \frac{1}{Z_1} \cos\theta_{T1} \quad ,$$

and the results look suggestive of matrix arithmetic:

$$\tilde{E}_{\|,1} = \tilde{E}_{\|,2} \cos\delta_1 - \tilde{H}_{\|,2} \frac{i \sin\delta_1}{Y_{1,TE}} \quad \text{and}$$

$$\tilde{H}_{\|,1} = -\tilde{E}_{\|,2} Y_{1,TE} i \sin\delta_1 + \tilde{H}_{\|,2} \cos\delta_1 \quad \text{, or}$$

$$\begin{bmatrix} \tilde{E}_{\|,1} \\ \tilde{H}_{\|,1} \end{bmatrix} = \begin{bmatrix} \cos\delta_1 & -i \sin\delta_1 / Y_{1,TE} \\ -iY_{1,TE} \sin\delta_1 & \cos\delta_1 \end{bmatrix} \begin{bmatrix} \tilde{E}_{\|,2} \\ \tilde{H}_{\|,2} \end{bmatrix} \equiv M_1 \begin{bmatrix} \tilde{E}_{\|,2} \\ \tilde{H}_{\|,2} \end{bmatrix} .$$

M_1 is called the **characteristic matrix** of layer 1.

Matrix formulation of the fields at the interfaces (continued)

- We could repeat this procedure for TM waves (see following slide), but it's so similar to what we just did that we'll just skip to the result:

$$\tilde{E}_{\parallel,1} = -\tilde{H}_{\parallel,2}\sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_{T1}\, i\sin\delta_1 + \tilde{E}_{\parallel,2}\cos\delta_1 \quad ,$$

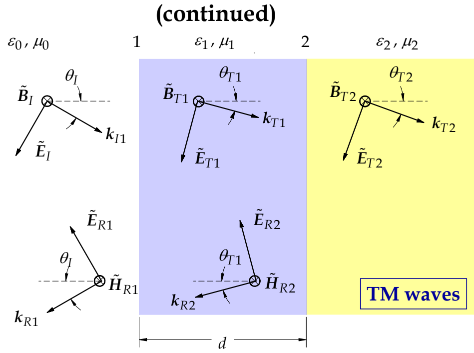
$$\tilde{H}_{\parallel,1} = \tilde{H}_{\parallel,2}\cos\delta_1 - \tilde{E}_{\parallel,2}\sqrt{\frac{\varepsilon_1}{\mu_1}}\frac{i\sin\delta_1}{\cos\theta_{T1}} \quad .$$

- Thus if we define

$$Y_{1,TM} = \sqrt{\frac{\varepsilon_1}{\mu_1}}\frac{1}{\cos\theta_{T1}} \quad ,$$

we get the same matrix equation as before, which we will write as:

Matrix formulation of the fields at the interfaces (continued)



Matrix formulation of the fields at the interfaces (continued)

$$\begin{bmatrix}\tilde{E}_{\parallel,1}\\ \tilde{H}_{\parallel,1}\end{bmatrix} = \begin{bmatrix}\cos\delta_1 & -i\sin\delta_1 / Y_1\\ -iY_1\sin\delta_1 & \cos\delta_1\end{bmatrix}\begin{bmatrix}\tilde{E}_{\parallel,2}\\ \tilde{H}_{\parallel,2}\end{bmatrix} \equiv M_1\begin{bmatrix}\tilde{E}_{\parallel,2}\\ \tilde{H}_{\parallel,2}\end{bmatrix} \quad .$$

- If there were yet a third surface to the right, the parallel components of the fields there could therefore be determined from

$$\begin{bmatrix}\tilde{E}_{\parallel,2}\\ \tilde{H}_{\parallel,2}\end{bmatrix} = M_2\begin{bmatrix}\tilde{E}_{\parallel,3}\\ \tilde{H}_{\parallel,3}\end{bmatrix} \quad ,$$

which can be combined with our first result to yield

$$\begin{bmatrix}\tilde{E}_{\parallel,1}\\ \tilde{H}_{\parallel,1}\end{bmatrix} = M_1 M_2\begin{bmatrix}\tilde{E}_{\parallel,3}\\ \tilde{H}_{\parallel,3}\end{bmatrix} \quad .$$

Matrix formulation of the fields at the interfaces (continued)

- And so on. Evidently, for a stack of p layers, the parallel components of the fields at the first and p+1th surface are related by

$$\begin{bmatrix} \tilde{E}_{\|,1} \\ \tilde{H}_{\|,1} \end{bmatrix} = M_1 M_2 \cdots M_p \begin{bmatrix} \tilde{E}_{\|,p+1} \\ \tilde{H}_{\|,p+1} \end{bmatrix} ,$$

and the whole stack can be said to have a characteristic matrix M given by

$$M = M_1 M_2 \cdots M_p = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} .$$